Subgroup Decomposition of the Gini Coefficient: A New Solution to an Old Problem*

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Abstract

We derive a novel decomposition of the Gini coefficient into within and between-group inequality terms that sum to the aggregate Gini coefficient. This decomposition is derived from a set of axioms that ensure desirable behavior for the within and between-group inequality terms. The decomposition of the Gini coefficient is unique given our axioms, easy to compute, and can be interpreted geometrically.

I Introduction

Empirical analyses of inequality often aim to quantify the contributions of within-group inequality and between-group differences to aggregate inequality. This classic objective is known as inequality decomposition by population subgroups. Examples include

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analyses of inequality within and between countries (Sala-i-Martin, 2006), demographic subgroups (Cowell and Jenkins, 1995), or firms (Song et al., 2019).

The Gini coefficient is the most popular measure of inequality, yet it lacks a universally accepted decomposition formula.[1] Existing approaches to decomposing the Gini coefficient are criticized for producing within-group or between-group inequality terms that behave counter-intuitively. Further, these approaches often rely on introducing a third term that describes neither within-group nor between-group inequality. This lack of a satisfactory subgroup decomposition formula is arguably the most significant drawback of the Gini coefficient, which is otherwise valued for its intuitive arithmetic definition and geometric relation to the Lorenz curve.

In this paper, we derive a novel decomposition formula for the Gini coefficient from a set of axioms that ensure desirable behavior for the within-group and between-group inequality terms. We show that these axioms uniquely determine the decomposition formula for the Gini coefficient. The decomposition is easy to compute, and has both a geometric and an arithmetic interpretation. We demonstrate the decomposition by analyzing the evolution of household income inequality within and between demographic subgroups in the United States over the past six decades.

Our paper contributes to the literature on decomposable inequality indices by proposing a standard for satisfactory decomposition that relies neither on the *aggregativity* requirement put forward by Bourguignon (1979) and Shorrocks (1980), nor the *path independence* requirement introduced by Foster and Shneyerov (2000). While aggregativity characterizes generalized entropy indices (but is violated by Foster-Shneyerov indices) and path independence characterizes Foster-Shneyerov indices (but is violated by generalized entropy indices), neither standard allows decomposition of the Gini coefficient. Our definition of decomposability accommodates both the generalized entropy and the Foster-Shneyerov indices, and also enables decomposition of the Gini coefficient.

II Notation and Definitions

We use $\mathbb{R}_+$ to denote the interval $[0, \infty)$ and $\mathbb{R}_{++}$ to denote the interval $(0, \infty)$. An inequality index $I\colon \mathcal{D} \to \mathbb{R}_+$ is a function that maps a space of distributions $\mathcal{D}$ to the non-negative real numbers and satisfies the five standard axioms of anonymity, scale independence, population independence, normalization, and the Pigou-Dalton principle

[1] The Gini coefficient is also one of the earliest measures of inequality, introduced by Corrado Gini in 1912. An overview of suggested decomposition formulas for the Gini coefficient can be found in Giorgi (2011).

of transfers.[2]

Throughout the paper, we represent income distributions by their generalized Lorenz curves.[3] A generalized Lorenz curve is an increasing convex function L: $[0, 1] \to \mathbb{R}_+$ with $L(0) = 0$.[4] The value of the generalized Lorenz curve at point $p \in [0, 1]$ is equal to p times the mean income of individuals below the pth quantile, and $L(1)$ is equal to the overall mean income. Hence, the generalized Lorenz curve is obtained by multiplying the standard Lorenz curve by the mean income and thus preserves the information on the mean of the distribution. In addition, generalized Lorenz curves provide a more general representation of distributions than cumulative distribution functions.[5] In particular, distributions with perfect inequality can be represented by a generalized Lorenz curve that is zero everywhere except at 1.

The Gini coefficient is defined for any generalized Lorenz curve L as[6]

$$G(L) = 2\int_0^1 p - \frac{L(p)}{L(1)} dp.$$

III Axiomatic Framework

We consider a population that is partitioned into K subgroups. Subgroup k's level of inequality, population size, and total income are denoted by I_k, n_k, and Y_k, respectively. We require that a subgroup decomposition of an inequality index I is the sum of a within-group inequality term W and a between-group inequality term B:

$$I = W + B.$$

We introduce axioms that satisfactory within-group and between-group inequality terms must satisfy. We call an inequality measure decomposable if it admits a decomposition

[2] Inclusion of the scale independence axiom means that we focus on indices of relative inequality rather than absolute inequality. Indices of relative inequality have the virtue of being independent of the unit of measurement.

[3] We frame our discussion in terms of income distributions, but income can be substituted with any non-negative real valued attribute.

[4] Note that by restricting generalized Lorenz curve to be increasing, we are ruling out negative incomes.

[5] For a distribution that has a cumulative distribution function, F, the generalized Lorenz curve can be expressed as $L(p) = \int_0^p F^{-1}(t)dt$, where F^{-1} is the generalized inverse of F (Gastwirth, 1971).

[6] If all incomes are zero, then the generalized Lorenz curve is a constant function at zero and the formula defining the Gini coefficient cannot be evaluated. We define the Gini coefficient to be zero in this case.

that satisfies these axioms.[7]

III.1 Within-Group Inequality

Within-group inequality summarizes how inequality within subgroups contributes to aggregate inequality. We require that within-group inequality depends only on subgroup inequality levels and aggregate characteristics, that is, each subgroup's total population and total income. Let $W_K\colon \mathbb{R}_+^{3K} \to \mathbb{R}_+$ denote within-group inequality for a population consisting of K subgroups. We posit the following axioms:

1. **Regularity.** W_K is continuous, and strictly increasing in I_k if $n_k, Y_k > 0$.
2. **Symmetry.** For any permutation P and for all $(I_k, n_k, Y_k)_{k=1}^K \in \mathbb{R}_+^{3K}$,
$$W_K\big((I_1, n_1, Y_1), \ldots, (I_K, n_K, Y_K)\big) = W_K\Big(P\big((I_1, n_1, Y_1), \ldots, (I_K, n_K, Y_K)\big)\Big).$$
3. **Scale and population independence.**
$$W_K\big((I_1, n_1, Y_1), \ldots, (I_K, n_K, Y_K)\big) = W_K\big((I_1, an_1, bY_1), \ldots, (I_K, an_K, bY_K)\big)$$
for all $a, b > 0$ and for all $(I_k, n_k, Y_k)_{k=1}^K \in \mathbb{R}_+^{3K}$.
4. **Normalization.** $W_K\big((0, n_1, Y_1), \ldots, (0, n_K, Y_K)\big) = 0$ for all $(n_k, Y_k)_{k=1}^K \in \mathbb{R}_+^{2K}$.
5. **Weak reflexivity.** $W_K\big((I, a_1 n, a_1 Y), \ldots, (I, a_K n, a_K Y)\big) = I$ for all $(I, n, Y) \in \mathbb{R}_{++}^3$ and for all $(a_k)_{k=1}^K \in \mathbb{R}_{++}^K$.
6. **Replacement.** For all $(I_k, n_k, Y_k)_{k=1}^K \in \mathbb{R}_+^{3K}$ and $m \leq K$,
$$\begin{aligned} &W_K\big((I_1, n_1, Y_1), \ldots, (I_K, n_K, Y_K)\big) \\ &= W_{K-m+1}\Big(\big(\tilde{I}, \textstyle\sum_{k=1}^m n_k, \sum_{k=1}^m Y_k\big), (I_{m+1}, n_{m+1}, Y_{m+1}), \ldots, (I_K, n_K, Y_K)\Big), \end{aligned}$$
where $\tilde{I} = W_m\big((I_1, n_1, Y_1), \ldots, (I_m, n_m, Y_m)\big)$.

Regularity ensures that within-group inequality increases continuously in subgroup inequality levels. Symmetry ensures that within-group inequality is independent of the labels given to each subgroup and is sometimes also called anonymity. Scale and

[7] In Online Appendix A, we discuss how our definition of decomposability relates to other standards suggested in the literature.

population independence ensures that the decomposition is independent of the size of the population and the unit of measurement for income. Normalization ensures that within-group inequality is zero when there is no inequality within any subgroup.

Weak reflexivity states that when all subgroups have the same level of inequality and mean income, within-group inequality is equal to the common inequality level across the subgroups. Hence, weak reflexivity ensures that within-group inequality equals aggregate inequality when the population consists of only one subgroup. Moreover, it ensures that within-group inequality does not change if that subgroup is divided into smaller subgroups with identical income distributions. Without weak reflexivity, within-group inequality would not be comparable across populations with different numbers of subgroups.

Finally, the replacement axiom states that the level of within-group inequality remains unchanged if we replace any number of subgroups by a single subgroup with population size and total income equal to the combined population and income of the replaced subgroups, and with inequality equal to the level of within-group inequality among the replaced subgroups. Together with symmetry, replacement ensures a consistent aggregation property. For example, consider computing within-group inequality among the fifty US states. Replacement guarantees that this can be done if we are given the level of within-group inequality in subaggregates, such as the eastern states and the western states, together with the population sizes and total incomes of these subaggregates. In particular, information on the inequality levels of individual states is not needed in this case. This is a natural property because the contribution to aggregate inequality stemming from inequality within individual states is already summarized in the levels of within-group inequality in the subaggregates.

Axioms 1 through 6 imply strong restrictions on the functional form that within-group inequality can have.

Theorem 1. *Let $(W_K)_{K=1}^{\infty}$ be a sequence of functions $W_K\colon \mathbb{R}_+^{3K} \to \mathbb{R}_+$. Then, (W_K) satisfies axioms 1-6 if and only if*

$$W_K\big((I_1, n_1, Y_1), \ldots, (I_K, n_K, Y_K)\big) = f^{-1}\Big(\sum_{k=1}^{K} \pi_k^{1-\alpha}\theta_k^{\alpha} f(I_k)\Big), \tag{1}$$

for all $K \geq 1$ and for all $(I_k, n_k, Y_k)_{k=1}^{K} \in \mathbb{R}_+^{3K}$, where f is some continuous and strictly increasing function with $f(0) = 0$, $\alpha \in \mathbb{R}$ is some real number, $\pi_k = n_k / \sum_{k=1}^{K} n_k$, and $\theta_k = Y_k / \sum_{k=1}^{K} Y_k$.

Theorem 1 holds that within-group inequality must take the form of a quasi-arithmetic mean of subgroup inequalities with weights that do not necessarily sum to one. Moreover, each subgroup's weight must itself be a weighted geometric average of the subgroup's population and income share. The proof proceeds by showing that, for given K and aggregate characteristics, the axioms imply that W_K satisfies bisymmetry. Bisymmetry has been used by Aczél (1948) and Münnich et al. (2000) to characterize quasi-arithmetic means. The rest of the proof shows that the generating function f and the form of the weight function do not depend on the number of subgroups or their aggregate characteristics. The complete proof is presented in appendix A.

The functional form in Theorem 1 nests the expressions for within-group inequality in the standard decomposition formulas for the generalized entropy indices and the Foster-Shneyerov indices – two important classes of decomposable inequality indices (Bourguignon, 1979; Shorrocks, 1980; Cowell, 1980; Foster and Shneyerov, 2000).[8] Interestingly, however, several decomposition formulas for the Gini coefficient suggested in the literature are ruled out by this theorem. For example, the most common decomposition formula for the Gini coefficient due to Bhattacharya and Mahalanobis (1967) is ruled out by Theorem 1 since the weight function is not a weighted geometric average, implying that the within-group inequality term violates weak reflexivity.[9] More recently, Shorrocks (2013) proposed an algorithm for inequality decomposition which results in a within-group inequality term that generally does not take the form of a quasi-arithmetic mean.

III.2 Between-Group Inequality

Between-group inequality summarizes how differences in income distributions across subgroups contribute to aggregate inequality. We define between-group inequality to be a function of subgroup income distributions, population sizes, and total incomes, that is, B_K: $(\mathcal{D} \times \mathbb{R}_+ \times \mathbb{R}_+)^K \to \mathbb{R}_+$. We posit the following axioms:

7. **Normalization.** If all subgroups have the same income distribution, then B_K is equal to zero.

[8]The within-group inequality term in the standard decomposition formula for the generalized entropy index of order α is given by $\sum_{k=1}^{K} \pi_k^{1-\alpha}\theta_k^{\alpha} I_k$, and the within-group inequality term in the standard decomposition formula for the Foster-Shneyerov index of order q is given by $\sum_{k=1}^{K} \pi_k I_k$.

[9]The within-group inequality term in the Gini decomposition of Bhattacharya and Mahalanobis (1967) is given by $\sum_{k=1}^{K} \pi_k \theta_k I_k$.

8. **Conditional distribution independence.** For given aggregate characteristics $(n_1, \ldots, n_K, Y_1, \ldots, Y_K) \in \mathbb{R}_+^{2K}$, if there exists a function $F\colon \mathbb{R}_+^K \to \mathbb{R}_+$ such that

$$I(L) = F\big(I_1, \ldots, I_K\big)$$

for all $(I_1, \ldots, I_K) \in \mathbb{R}_+^K$, then B_K does not depend on the distribution of income within subgroups.

Normalization ensures that between-group inequality is zero when the subgroups' income distributions are identical so that aggregate inequality must be entirely due to income differences within subgroups.[10]

Conditional distribution independence generalizes a property imposed by Bourguignon (1979) and Shorrocks (1980), which requires that between-group inequality is unaffected by income transfers within subgroups. These authors impose distribution independence in the context of aggregative inequality indices, which are indices for which aggregate inequality is a function of subgroup population sizes, inequality levels, and average incomes. It is natural to require distribution independence for aggregative inequality indices since the impact of within-group transfers on aggregate inequality is fully summarized by changes in subgroup inequality levels and should therefore be attributed to within-group inequality.

However, imposing unconditional distribution independence on nonaggregative inequality indices like the Gini coefficient is inappropriate, as within-group transfers only keep the means constant while aggregate inequality is generally also sensitive to whether these transfers make the subgroup distributions more or less similar in other moments. Clearly, changes in aggregate inequality due to subgroup distributions becoming more or less similar should be captured by between-group inequality, which therefore cannot be distribution independent.

Instead, *conditional* distribution independence only imposes distribution independence whenever, for given aggregate characteristics, the aggregate inequality index is a function of subgroup inequality indices alone. In these special cases, the inequality index is aggregative and therefore between-group inequality should indeed be independent of the income distribution within subgroups. As is well known, the Gini coefficient

[10]Note that Axiom 7 is implied by Axiom 5 (Weak reflexivity) and the requirement that aggregate inequality is the sum of within and between-group inequality. In online appendix A, we strengthen Axiom 7 to show that a similar axiomatic framework can be used to derive unique decompositions not only for the Gini coefficient but also other classes of inequality indices. The strengthened version of Axiom 7 is not implied by Axiom 5 and is satisfied by the Gini decomposition derived in this paper.

is aggregative when all but one subgroup have zero income or population share and the aggregate characteristics are therefore such that the subgroup income distributions cannot overlap.

IV Decomposition of the Gini Coefficient

We next show that there exists a unique decomposition of the Gini coefficient into a within-group and a between-group term that satisfies all the axioms introduced in section III.

IV.1 Decomposition Formula

Theorem 2. *A decomposition for the Gini coefficient satisfies axioms 1-8 if and only if the within-group inequality term is*

$$G^W = \left(\sum_{k=1}^{K} \sqrt{\pi_k \theta_k G_k} \right)^2, \tag{2}$$

where π_k, θ_k, and G_k are the population share, income share, and Gini coefficient of subgroup k, and the between-group inequality term is

$$G^B = \frac{1}{2\mu} \left(\Theta - \sum_{k \neq l} \pi_k \pi_l \sqrt{\Delta_k \Delta_l} \right), \tag{3}$$

where μ is the aggregate mean, $\Theta = \mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}} |y_i - y_j|\big]$ is the cross mean absolute difference, $\Delta_k = \mathbb{E}\big[|y_i - y_j| \big| g_i = g_j = k\big]$ is the mean absolute difference within subgroup k, and y_i and g_i denote the income and subgroup affiliation of individual i.

Within-group inequality in the Gini decomposition is a weighted power mean of subgroup inequalities where each subgroup is weighted by the geometric mean of its income and population share.[11] Between-group inequality is the difference between the cross mean absolute difference, defined as the mean absolute difference of individuals drawn from different subgroups, and a weighted sum of geometric averages of subgroup mean absolute differences, divided by twice the aggregate mean. In practice, however,

[11] A power mean, also known as generalized mean or Hölder mean, with exponent p is a function $M_p(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}}$ and includes as special cases the arithmetic, geometric, and harmonic means.

it is generally easier to compute the between-group inequality term as a residual, $G^B = G - G^W$.

The proof of Theorem 2 proceeds by first showing that the expressions for G^W and G^B sum to the aggregate Gini coefficient and that the decomposition satisfies axioms 1-8. We then show uniqueness by leveraging the fact that the Gini coefficient is aggregative when subgroup income distributions cannot overlap. In these special cases, Axiom 8 together with Theorem 1 pin down the generating function f and the weight structure in equation (1), which uniquely determine the within-group and between-group inequality terms. The complete proof is presented in appendix A.

In online appendix A, we show that our axiomatic framework can also be used to uniquely determine the standard decomposition formulas for the generalized entropy and Foster-Shneyerov indices. In online appendix B, we derive asymptotic confidence intervals for the within and between-group inequality terms of the Gini decomposition. In online appendix C, we show how the decomposition formula can be extended to the multivariate Gini coefficients introduced in Koshevoy and Mosler (1997).

IV.2 Discussion of the Gini Decomposition

The within and between-group inequality terms in the Gini decomposition of Theorem 2 satisfy several additional properties that we have not directly imposed as axioms. First, as a weighted power mean the within-group inequality term is homogeneous.[12] That is, any redistribution of incomes within subgroups that reduces subgroup Gini coefficients by some given factor will also reduce within-group inequality by the same factor. Note, however, that the within-group inequality term is not linear in the subgroup Gini coefficients.

Second, the between-group inequality term summarizes the contribution to aggregate inequality stemming from differences in the means and the shapes of the subgroup income distributions. Unlike the between-group inequality term for the Theil index (and other generalized entropy indices) that depends only on differences in subgroup means, the between-group inequality term for the Gini coefficient depends on differences in all moments of the subgroup income distributions. This is natural since, unlike other inequality indices, the aggregate Gini coefficient itself depends on all the moments of the subgroup income distributions. Specifically, Proposition 1 shows that changes in the subgroup income distributions can affect the aggregate Gini coefficient even if all

[12] Homogeneity is in fact the only additional property that power means have over quasi-arithmetic means implied by Theorem 1.

subgroup Gini coefficients as well as any number of subgroup moments are held fixed.[13]

Proposition 1. *The aggregate Gini coefficient cannot be written as a function of subgroup population sizes, Gini coefficients, and any number of subgroup moments. That is, there does not exist a set of moments* Ω *and a function* F*, such that*

$$G(L) = F\big(G_1, \ldots, G_K; \Omega_1, \ldots, \Omega_K; n_1, \ldots, n_K\big), \tag{4}$$

where Ω_k *is the vector of moments for subgroup* k*.*

How between-group inequality depends on differences in the shapes of the subgroup income distributions is also intuitive. First, as is shown in Proposition 8 of section IV.3, the between-group inequality term is zero if and only if the distribution of income is identical across subgroups.

Second, the between-group inequality term becomes smaller as subgroup income distributions become more similar. Specifically, Proposition 2 states that between-group inequality is reduced when the subgroup affiliations for a random subset of individuals are permuted so that subgroup income distributions become unambiguously more similar to each other.[14] Moreover, the only case in which this operation does not make subgroup income distributions more similar is when the distributions are identical to begin with. Because the aggregate Gini coefficient is sensitive to all moments of the subgroup income distributions, this is the only case in which between-group is not reduced.

Proposition 2. *In an infinite population, randomly permuting the subgroup affiliations for a subset of randomly selected individuals with positive population share weakly reduces between-group inequality (while keeping the aggregate Gini coefficient constant). Between-group inequality remains constant if and only if all subgroups have the same distribution of income.*

Similarly, merging several subgroups into one should decrease between-group inequality as any differences between the merged subgroups can no longer contribute to overall between-group inequality. Proposition 3 states that this operation indeed reduces between-group inequality unless the income distributions of the merged subgroups are all identical.

[13] The claim that the between-group inequality term in the Gini decomposition is sensitive to any differences in subgroup income distributions is a corollary of Proposition 1.

[14] Equivalently, one may consider replacing the incomes of a fixed fraction of randomly sampled individuals in each subgroup with a random draw from the aggregate income distribution. Equivalency between these two operations is a direct consequence of the anonymity property of inequality indices.

Proposition 3. *Merging any $m \leq K$ subgroups into one subgroup weakly reduces the between-group inequality term in the Gini decomposition. Between-group inequality is unaffected if and only if the merged subgroups have identical income distributions.*

There also exist operations that reduce both within-group inequality and between-group inequality. For example, redistributing incomes so that the difference between each individual's income and the average income is reduced by a given percentage clearly reduces aggregate inequality.[15] Moreover, as such redistribution at the same time compresses subgroup income distributions and brings them closer to each other, one may expect it to decrease within-group and between-group inequality by similar proportions. Proposition 4 shows that this is indeed the case.

Proposition 4. *Let y_i denote the income of individual i, and let μ denote the average income in the population. For some $\alpha \in [0, 1]$, replacing every income y_i by $\tilde{y}_i = y_i - \alpha(y_i - \mu)$ reduces within-group inequality and between-group inequality in the Gini decomposition by a fraction α.*

It is often of interest to know by how much aggregate inequality would be reduced if all subgroup means were equalized while keeping the level of within-group inequality fixed. For many inequality indices this question is not easily answered. For example, as was pointed out by Shorrocks (1980), there is no obvious operation for the Theil index that would eliminate differences in average incomes between subgroups and also keep the within-group inequality term fixed.[16] Proposition 5 shows that the decomposition of the Gini coefficient admits an operation that can be used to eliminate differences between subgroups in average incomes while keeping within-group inequality fixed. This property can be used to further decompose between-group inequality into a first part that reflects differences in means and a second part that reflects differences in the shape of the distribution. We implement such an exercise in online appendix F.

Proposition 5. *Lump-sum transfers between subgroups do not affect within-group inequality in the Gini decomposition.*

Finally, we note that scaling or translating all incomes does not affect the share of aggregate inequality that is attributed to within-group or between-group inequality

[15]This can be achieved with a flat tax and a lump sum transfer, which is the standard setup in the optimal linear tax literature (see, e.g., Piketty and Saez (2013)), in the absence of behavioral responses to the tax.

[16]For example, scaling the incomes in each subgroup so that the subgroup means are equalized changes the subgroup income shares and thereby affects the within-group inequality term in the decomposition of the Theil index. The only generalized entropy index for which scaling incomes does not affect within-group inequality is the mean log deviation.

in the Gini decomposition. While the scale independence of the decomposition follows directly from the scale independence of the Gini coefficient and therefore applies to the decomposition of all scale-independent inequality indices, translation independence is a special feature of the Gini decomposition. Translation independence is convenient if one wants to decompose the Gini coefficient of attributes that can be negative, such as wealth.

Proposition 6. *The Gini decomposition is both scale and translation invariant. Specifically, changing incomes for each individual i from y_i to $\tilde{y}_i = ay_i + b$ does not affect the relative magnitude of within-group and between-group inequality for any $a > 0$ and $b \in \mathbb{R}$ such that each subgroup's mean income remains positive.*

IV.3 Geometric Interpretation

The Gini coefficient is traditionally defined as twice the area between the Lorenz curve and the line of perfect equality. To derive a geometric interpretation for the Gini decomposition, it is useful to define the *Lorenz region* as the region within the unit square that is bounded by the Lorenz curve and its centrally reflected counterpart (see Figure 1).[17] Clearly, the Gini coefficient is equal to the area of the Lorenz region.

For a given partition of the population into subgroups, we can also define *subgroup Lorenz regions* by scaling each subgroup's Lorenz region with a vector (π_k, θ_k) where π_k and θ_k are subgroup k's population and income shares, respectively (see Figure 1). Importantly, Zagier (1983) shows that there is a geometric relationship between the aggregate and the subgroup Lorenz regions. Specifically, Proposition 7 (proven as part of Theorem 1 in Zagier (1983)) states that the Lorenz region of the aggregate population is the Minkowski sum of the subgroup Lorenz regions.[18]

Proposition 7. *A population consisting of $K \geq 2$ subgroups with subgroup Lorenz regions $\Lambda_1, \Lambda_2, \ldots, \Lambda_K$ has an aggregate Lorenz region*

$$\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \ldots \oplus \Lambda_K,$$

where $\oplus$ denotes the Minkowski sum of sets.

Figure 1 illustrates the Minkowski addition of two subgroup Lorenz regions in the case

[17] This region is also known as the Lorenz zonoid, see Koshevoy and Mosler (1997).

[18] The Minkowski sum, also known as the vector sum, of two sets A and B is defined as $A \oplus B = \{a + b : a \in A, b \in B\}$.

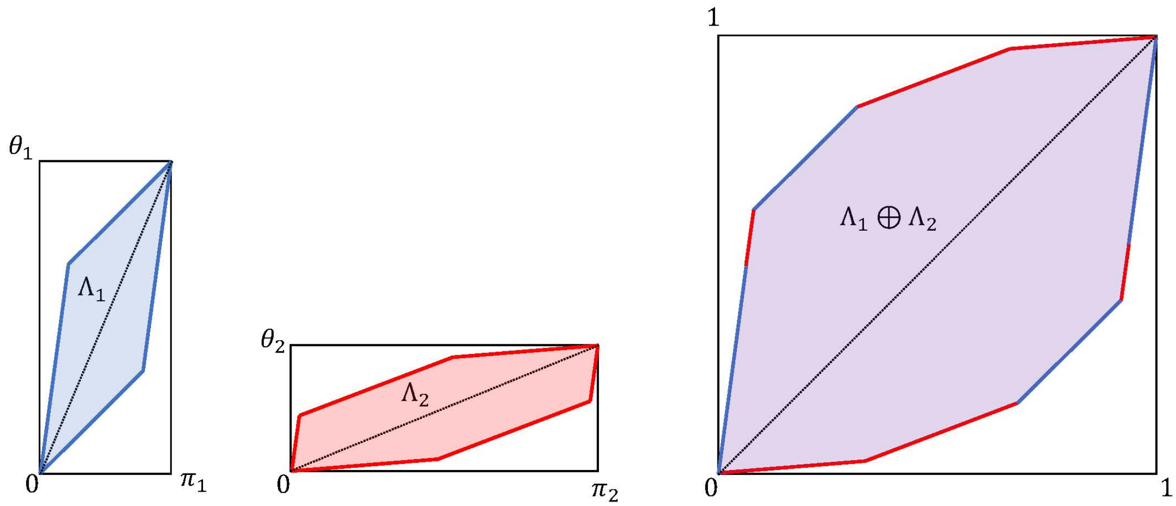

Figure 1: Aggregation of subgroup Lorenz regions. In this example, there are two different levels of income in subgroup 1 and three different levels of income in subgroup 2. In the aggregate Lorenz curve, individuals from both subgroups are ordered by their income. Geometrically, this corresponds to arranging the linear segments of the subgroup Lorenz curves in ascending order by their slopes. The resulting aggregate Lorenz region coincides with the Minkowski sum of the subgroup Lorenz regions.

of discrete income distributions. In online appendix E, we illustrate the Minkowski addition in the case of continuous income distributions.

A useful implication of representing the aggregate Gini coefficient as the area of the Minkowski sum of subgroup Lorenz regions is that we can make use of an important result relating the areas of compact sets: the Brunn-Minkowski theorem. Specifically, the Brunn-Minkowski theorem provides a lower bound for the aggregate Gini coefficient in terms of subgroup Gini coefficients and aggregate characteristics.

Proposition 8 (Brunn-Minkowski theorem)**.** *For a population with Lorenz region Λ and Gini coefficient G consisting of K subgroups with subgroup Lorenz regions $\Lambda_1, \ldots, \Lambda_K$, Gini coefficients $G_1, \ldots, G_K$, population shares $\pi_1, \ldots, \pi_K$, and income shares $\theta_1, \ldots, \theta_K$, we have*

$$G = |\Lambda| \geq \Big(\sum_{k=1}^{K} \sqrt{|\Lambda_k|} \Big)^2 = \Big(\sum_{k=1}^{K} \sqrt{\pi_k \theta_k G_k} \Big)^2 = G^W$$

where $|\cdot|$ is the Lebesgue measure. The inequality holds as equality if and only if $\Lambda_1, \ldots, \Lambda_K$ are homothetic.

In light of Proposition 8, within-group inequality can be interpreted geometrically as the minimal area of the aggregate Lorenz region for given areas of the subgroup

Lorenz regions. Similarly, between-group inequality is the excess area in the aggregate Lorenz region that is not explained by the areas of the subgroup Lorenz regions.

As the Minkowski sum of the subgroup Lorenz regions, the area of the aggregate Lorenz region depends both on the areas of the subgroup Lorenz regions as well as on how similar the shapes of the subgroup Lorenz regions are. The constrained minimum is achieved when the subgroup income distributions are all identical so that the subgroup Lorenz regions are homothetic. As a consequence, between-group inequality measures the excess area in the population Lorenz region resulting from non-homotheticity of the subgroup Lorenz regions. In other words, between-group inequality measures how differences in the shapes of the subgroup income distributions contribute to aggregate inequality.[19] In online appendix D, we offer an arithmetic interpretation for the Gini decomposition.

The fact that the expression for the within-group inequality term is a lower bound for the aggregate Gini coefficient has previously been shown in Zagier (1983) using the Brunn-Minkowski inequality. Zagier (1983) studies the problem of bounding the aggregate Gini coefficient for given subgroup Gini coefficients, means, and population shares, and derives our expression for the within-group inequality term as one of several lower bounds for the aggregate Gini coefficient.[20] The paper also notes that aggregate inequality is smaller when subgroups are more similar in their income distributions. Because the paper considers the situation where subgroup Gini coefficients and aggregate characteristics are fixed, this is equivalent to noting that our between-group inequality term is smaller when the subgroup income distributions are more similar.

IV.4 Subgroup Consistency

The Gini decomposition together with its geometric interpretation provides helpful insights into a notorious behavior of the Gini coefficient that is called *subgroup inconsistency*; an increase in inequality within subgroups, while keeping subgroup means and

[19]In convex geometry, the excess area of the Minkowski sum relative to the Brunn-Minkowski lower bound is sometimes called the Brunn-Minkowski deficit, which has been shown to relate to the symmetry of the added sets. Figalli et al. (2009), for example, show that the Brunn-Minkowski deficit is bounded from below by an increasing function of the relative asymmetry of the added sets.

[20]Note that the within-group inequality term for the Gini coefficient is not generally the best lower bound in Zagier's setup. Unlike in our setup where only the areas of the subgroup Lorenz regions (the product of π_k, θ_k, and G_k) are given, Zagier looks for the best lower bound for the aggregate Gini coefficient when the subgroup Gini coefficients, population shares, and means are given individually. The implied best lower bound therefore exceeds our within-group inequality term as it also incorporates the part of between-group inequality stemming from differences in subgroup means.

population sizes constant, can lead to a decrease in aggregate inequality.[21] Subgroup consistency, which rules out this behavior, has been proposed as a requirement for decomposability (Cowell, 2000). The results in this paper clarify that the Gini coefficient can violate subgroup consistency if an increase in inequality within subgroups also makes the subgroup income distributions more similar so that between-group inequality decreases more than within-group inequality increases.

Moreover, we note that the Gini coefficient does satisfy a weaker version of subgroup consistency. In particular, Proposition 9 states that any transfers within subgroups that increase subgroup inequalities according to the Lorenz criterion must also increase the aggregate Gini coefficient.[22] It follows that if transfers that increase inequality within subgroups reduce the aggregate Gini coefficient, it must be the case that for at least one subgroup the new Lorenz curve intersects with the old Lorenz curve. The proof of Proposition 9 makes use of the fact that outward shifts of the subgroup Lorenz curves must result in an outward shift of the aggregate Lorenz curve.

Proposition 9 (Weak subgroup consistency)**.** *Transfers within one or more subgroups that increase the level of subgroup inequality according to the Lorenz criterion also increase aggregate inequality.*

V Empirical Application

We demonstrate our decomposition of the Gini coefficient by analyzing the extent to which household income inequality in the United States reflects inequality within versus between demographic subgroups. In this decomposition, between-group inequality summarizes how much of overall inequality is "explained" by differences in income distributions between demographic subgroups, while within-group inequality is a measure of "residual inequality" (Cowell and Jenkins, 1995; Juhn et al., 1993).[23]

We use data on household income in the United States from the Current Population Surveys and define demographic subgroups by the age, education, sex, and race of the

[21] Different examples of specific distributions and transfers that produce this behavior have been discussed in the literature (see, for example, Cowell (1988)).

[22] The Lorenz criterion states that inequality of distribution A exceeds that of distribution B if the Lorenz curve of A is always below that of B, and therefore any inequality index must judge inequality to be higher in distribution A than in distribution B.

[23] Cowell and Jenkins (1995) study how much of aggregate inequality in the United States can be "explained" by demographic characteristics using different Atkinson indices. Juhn et al. (1993) document rising wage inequality among men who are otherwise similar in terms of education and labor market experience. Since then, a large literature in labor economics devoted to explaining the rise in "residual inequality" has emerged.

Figure 2: Household-level income inequality within and between demographic subgroups. Households are grouped by the age ($\leq$ 35, 36–45, 46-55, 56-65,$>$ 65), education (no college, some college, Bachelor's degree, more than Bachelor's degree), sex (male, female), and race (Black, White, other) of the household head. Asymptotic confidence intervals are computed using the formulas derived in Online Appendix B.

household head.[24] Figure 2 shows the evolution of within-group and between-group inequality for the years 1967–2021. Overall, and in line with the conclusion in Cowell and Jenkins (1995), a relatively minor share of aggregate inequality is explained by differences in income distributions between demographic subgroups. Moreover, over the past five decades the aggregate Gini coefficient has increased from 0.39 to 0.50, and this increase was clearly driven by a rise in within-group inequality, which has increased from 0.29 to 0.41. At the same time, between-group inequality has decreased slightly from 0.10 to 0.09. As a consequence, the share of aggregate inequality that can be attributed to demographic characteristics has decreased from 25 to 17 percent. In online appendix F, we show how the fact that lump-sum transfers between subgroups do not affect within-group inequality can be used to isolate the part of between-group

[24]To deal with topcoding in the CPS data, we follow Heathcote et al. (2023).

inequality that is due to differences in average earnings between subgroups.[25]

VI Concluding Remarks

The Gini coefficient is the most prominent measure of inequality. Yet, there has been much disagreement regarding its decomposition by population subgroups. As a consequence, researchers often rely on other inequality indices when assessing the contribution of within-group and between-group inequality to aggregate inequality, even if they would otherwise prefer to work with the Gini coefficient.

In this paper, we show that the Gini coefficient admits a satisfactory decomposition formula derived from a set of axioms that ensure desirable behavior for the within-group and between-group inequality terms. The decomposition is novel, unique given our axioms, and easy to compute. Moreover, it can be interpreted both geometrically and arithmetically. Given these advantages, the Gini decomposition derived in this paper can be a valuable tool for empirical research.

A Proofs

A.1 Proof of Theorem 1

It is easy to verify that the functional form in (1) satisfies axioms 1-6. The remainder of the proof shows that if (W_K) satisfies axioms 1-6, then (1) holds. We first show that for given $K \geq 2$ and aggregate characteristics $(n_k, Y_k)_{k=1}^K \in \mathbb{R}^{2K}_{++}$, W_K must have the form

$$W_K\big((I_1, n_1, Y_1), \ldots, (I_K, n_K, Y_K)\big) = f^{-1}\Bigg(\sum_{k=1}^{K} a_k f(I_k)\Bigg), \tag{5}$$

for all $(I_k)_{k=1}^K \in \mathbb{R}^K_+$, where $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous increasing function such that $f(0) = 0$ and $(a_k)_{k=1}^K \in \mathbb{R}^K_{++}$.

Suppose $K \geq 2$ and $(n_k, Y_k)_{k=1}^K \in \mathbb{R}^{2K}_{++}$ are fixed and let $B(x_1, \ldots, x_K) =$

[25] In the same online appendix, we also implement an additional empirical application focusing on gender earnings inequality in the United States. Both of our empirical applications apply the decomposition formula to study inequality in a single country over time. Alternatively, one could use the decomposition to compare inequality across countries. For example, if we observe that country A has a higher Gini coefficient than country B, and if the within-group inequality term is also higher in country A than in country B, then we can conclude that country A is more unequal, in part, because inequality within the subgroups contribute more to aggregate inequality in country A than in country B.

$W_K\big((x_1,n_1,Y_1),\dots,(x_K,n_K,Y_K)\big)$. Now, B satisfies the bisymmetry equation

$$B\big(B(x_{11},\dots,x_{1K}),\dots,B(x_{K1},\dots,x_{KK})\big)=B\big(B(x_{11},\dots,x_{K1}),\dots,B(x_{1K},\dots,x_{KK})\big)$$

for all $(x_{ij})\in\mathbb{R}_+^{K\cdot K}$. To show this, we use axioms 2, 3, and 6:

$$\begin{aligned}
&B(B(x_{11},\dots,x_{1K}),\dots,B(x_{K1},\dots,x_{KK}))\\
=&W_K\big((W_K((x_{11},n_1,Y_1),\dots,(x_{1K},n_K,Y_K)),n_1,Y_1),\dots\\
&\dots,(W_K((x_{K1},n_1,Y_1),\dots,(x_{KK},n_K,Y_K)),n_K,Y_K)\big)\\
=&W_K\big((W_K((x_{11},\tfrac{n_1}{\sum_{k=1}^K n_k}n_1,\tfrac{Y_1}{\sum_{k=1}^K Y_k}Y_1),\dots,(x_{1K},\tfrac{n_1}{\sum_{k=1}^K n_k}n_K,\tfrac{Y_1}{\sum_{k=1}^K Y_k}Y_K)),n_1,Y_1),\\
&\dots,(W_K((x_{K1},\tfrac{n_K}{\sum_{k=1}^K n_k}n_1,\tfrac{Y_K}{\sum_{k=1}^K Y_k}Y_1),\dots,(x_{KK},\tfrac{n_K}{\sum_{k=1}^K n_k}n_K,\tfrac{Y_K}{\sum_{k=1}^K Y_k}Y_K)),n_K,Y_K)\big)\\
=&W_{K^2}\big((x_{11},\tfrac{n_1}{\sum_{k=1}^K n_k}n_1,\tfrac{Y_1}{\sum_{k=1}^K Y_k}Y_1),\dots,(x_{1K},\tfrac{n_1}{\sum_{k=1}^K n_k}n_K,\tfrac{Y_1}{\sum_{k=1}^K Y_k}Y_K),\\
&\dots,(x_{K1},\tfrac{n_K}{\sum_{k=1}^K n_k}n_1,\tfrac{Y_K}{\sum_{k=1}^K Y_k}Y_1),\dots,(x_{KK},\tfrac{n_K}{\sum_{k=1}^K n_k}n_K,\tfrac{Y_K}{\sum_{k=1}^K Y_k}Y_K)\big)\\
=&W_{K^2}\big((x_{11},\tfrac{n_1}{\sum_{k=1}^K n_k}n_1,\tfrac{Y_1}{\sum_{k=1}^K Y_k}Y_1),\dots,(x_{K1},\tfrac{n_K}{\sum_{k=1}^K n_k}n_1,\tfrac{Y_K}{\sum_{k=1}^K Y_k}Y_1),\\
&\dots,(x_{1K},\tfrac{n_1}{\sum_{k=1}^K n_k}n_K,\tfrac{Y_1}{\sum_{k=1}^K Y_k}Y_K),\dots,(x_{KK},\tfrac{n_K}{\sum_{k=1}^K n_k}n_K,\tfrac{Y_K}{\sum_{k=1}^K Y_k}Y_K)\big)\\
=&W_K\big((W_K((x_{11},\tfrac{n_1}{\sum_{k=1}^K n_k}n_1,\tfrac{Y_1}{\sum_{k=1}^K Y_k}Y_1),\dots,(x_{K1},\tfrac{n_K}{\sum_{k=1}^K n_k}n_1,\tfrac{Y_K}{\sum_{k=1}^K Y_k}Y_1)),n_1,Y_1),\\
&\dots,(W_K((x_{1K},\tfrac{n_1}{\sum_{k=1}^K n_k}n_K,\tfrac{Y_1}{\sum_{k=1}^K Y_k}Y_K),\dots,(x_{KK},\tfrac{n_K}{\sum_{k=1}^K n_k}n_K,\tfrac{Y_K}{\sum_{k=1}^K Y_k}Y_K),n_K,Y_K)\big)\\
=&W_K\big((W_K((x_{11},n_1,Y_1),\dots,(x_{K1},n_K,Y_K)),n_1,Y_1),\\
&\dots,(W_K((x_{1K},n_1,Y_1),\dots,(x_{KK},n_K,Y_K)),n_K,Y_K)\big)\\
=&B(B(x_{11},\dots,x_{K1}),\dots,B(x_{1K},\dots,x_{KK})).
\end{aligned}$$

Thus, B is a continuous function that is strictly increasing in each of its arguments, symmetric, and satisfies bisymmetry. As shown in Aczél (1948), B can be used to construct a function M: $\mathbb{R}_+^K\to\mathbb{R}_+$ that has the same properties but is also reflexive by defining $M(x_1,\dots,x_K)=F^{-1}(B(x_1,\dots,x_K))$, where $F(z)=B(z,\dots,z)$.[26] Hence, M satisfies the conditions of Theorem 2 in Münnich et al. (2000), which states that

$$M(x_1,\dots,x_K)=f^{-1}\left(\sum_{k=1}^K b_k f(x_k)\right), \tag{6}$$

for some continuous increasing function f: $\mathbb{R}_+\to\mathbb{R}_+$ such that $f(0)=0$ and for some

[26] Aczél (1948) shows this in the case where B is a bivariate function but the same proof generalizes to the case of K variables.

$b_k > 0$ such that $\sum_{k=1}^K b_k = 1$. Aczél (1948) shows that (6) implies

$$B(x_1, \dots, x_K) = f^{-1}\bigg(\sum_{k=1}^{K} a_k f(x_k) + b \bigg),$$

where $a_k = ab_k$ for some $a \neq 0$ and $b \in \mathbb{R}$. Since $f(0) = 0$, then by the normalization axiom, $B(0, \dots, 0) = f^{-1}(b) = 0$, which implies $b = 0$. Thus, we get equation (5).

In equation (5), the constants a_k and the generating function f can depend on K and $(n_k, Y_k)_{k=1}^K$. Note that f and cf generate the same W_K for any constant $c > 0$. Thus, if $g(x) = cf(x)$ for all x, we call f and g the same generating function. We next show that the generating function of W_K is independent of K and $(n_k, Y_k)_{k=1}^K$.

Let $K \geq 4$. By replacement and symmetry, we have

$$\begin{aligned} &W_K\big((I_1, n_1, Y_1), \dots, (I_K, n_K, Y_K)\big) \\ &= W_3\Bigg((I_1, n_1, Y_1), (I_2, n_2, Y_2), \Bigg(W_{K-2}\big((I_3, n_3, Y_3), \dots, (I_K, n_K, Y_K)\big), \sum_{k=3}^{K} n_k, \sum_{k=3}^{K} Y_k\Bigg)\Bigg) \end{aligned}$$

for all $(I_k, n_k, Y_k)_{k=1}^K \in \mathbb{R}_+^{3K}$. Suppose $(n_k, Y_k)_{k=1}^K \in \mathbb{R}_{++}^{2K}$ are given. By substituting equation (5) to the both sides of the above equation, we get

$$f^{-1}\bigg(\sum_{k=1}^{K} a_k f(I_k) \bigg) = g^{-1}\Bigg(b_1 g(I_1) + b_2 g(I_2) + b_3 g\Bigg(h^{-1}\bigg(\sum_{k=3}^{K} c_k h(I_k) \bigg) \Bigg) \Bigg), \tag{7}$$

where f, g, h are continuous and strictly increasing functions with $f(0) = g(0) = h(0) = 0$. Now, set $I_k = 0$ for $k = 3, \dots, K$ and define $x = f(I_1), y = f(I_2)$. Then, (7) implies

$$\phi(a_1 x + a_2 y) = b_1 \phi(x) + b_2 \phi(y)$$

for all $x, y \in \mathbb{R}_+$, where $\phi = g \circ f^{-1}$. By Theorem 2 on page 67 in Aczél (1966), this equation has a solution only if $a_1 = b_1$ and $a_2 = b_2$ and the solution is $\phi(x) = cx$ for some constant $c \neq 0$. Thus $g \circ f^{-1}(x) = cx$ which implies $g(x) = cf(x)$ for all $x \in \mathbb{R}_+$, i.e., W_K and W_3 have the same generating function for any $K \geq 4$.

Inserting this result into equation (7) when $K = 4$, we get

$$f^{-1}\bigg(\sum_{k=1}^{4} a_k f(I_k) \bigg) = f^{-1}\Bigg(a_1 f(I_1) + a_2 f(I_2) + b_3 f\Bigg(h^{-1}\bigg(c_1 h(I_3) + c_2 h(I_4) \bigg) \Bigg) \Bigg).$$

By defining $x = h(I_3)$ and $y = h(I_4)$, we can rewrite above equation as

$$\frac{a_3}{b_3}\phi(x) + \frac{a_4}{b_3}\phi(y) = \phi\big(c_1 x + c_2 y\big)$$

for all $x, y \in \mathbb{R}_+$, where $\phi = f \circ h^{-1}$. This yields $c_1 = a_3/b_3$, $c_2 = a_4/b_3$ and $\phi(x) = f \circ h^{-1}(x) = cx$ for some constant $c \neq 0$, or $f(x) = ch(x)$ for all x. Thus, W_2 and W_4 have the same generating function. Hence, we get that the generating function of W_K is independent of K for $K \geq 2$. Moreover, since aggregate characteristics $(n_k, Y_k)_{k=1}^K$ were arbitrary, the generating function does not depend on them.

Finally, we show that the constants a_k in (5) have the form $a_k = \pi_k^{1-\alpha}\theta_k^{\alpha}$, where $\pi_k = n_k / \sum_{k=1}^K n_k$ and $\theta_k = Y_k / \sum_{k=1}^K Y_k$. Using replacement and equation (5), we get

$$f^{-1}\left(\sum_{k=1}^{K} a_k f(I_k)\right) = f^{-1}\left(b_1 \sum_{k=1}^{K-1} c_k f(I_k) + b_2 f(I_K)\right) \tag{8}$$

for all $K \geq 2$, $(I_k)_{k=1}^K \in \mathbb{R}_+^K$ and $(n_k, Y_k)_{k=1}^K \in \mathbb{R}_{++}^{2K}$, where $a_k = a(n_k, Y_k; n_1, Y_1, \ldots, n_K, Y_K)$, $c_k = c(n_k, Y_k; n_1, Y_1, \ldots, n_{K-1}, Y_{K-1})$, $b_1 = b\big(\sum_{k=1}^{K-1} n_k, \sum_{k=1}^{K-1} Y_k; \sum_{k=1}^{K-1} n_k, \sum_{k=1}^{K-1} Y_k, n_K, Y_K\big)$, and $b_2 = b\big(n_K, Y_K; \sum_{k=1}^{K-1} n_k, \sum_{k=1}^{K-1} Y_k, n_K, Y_K\big)$ for some functions a, b, and c. By setting $I_k = 0$ for $k = 1, \ldots, K-1$, we get

$$a(n_K, Y_K; n_1, Y_1, \ldots, n_K, Y_K) = b\left(n_K, Y_K; \sum_{k=1}^{K-1} n_k, \sum_{k=1}^{K-1} Y_k, n_K, Y_K\right)$$

for all $(n_k, Y_k)_{k=1}^K \in \mathbb{R}_+^{2K}$. By scale and population independence, we get

$$a(n_K, Y_K; n_1, Y_1, \ldots, n_K, Y_K) = b(\pi_K, \theta_K; 1-\pi_K, 1-\theta_K, \pi_K, \theta_K) = w(\pi_K, \theta_K) \tag{9}$$

for some function w. Due to symmetry, this result holds for any k. Moreover, since K is arbitrary, same result applies to functions b and c. By substituting (9) into equation (8) and setting $I_j = 0$ for all $j \neq k$ for some $k = 1, \ldots, K-1$, we get

$$w\left(\frac{n_k}{\sum_{k=1}^K n_k}, \frac{Y_k}{\sum_{k=1}^K Y_k}\right) = w\left(\frac{\sum_{k=1}^{K-1} n_k}{\sum_{k=1}^K n_k}, \frac{\sum_{k=1}^{K-1} Y_k}{\sum_{k=1}^K Y_k}\right) w\left(\frac{n_k}{\sum_{k=1}^{K-1} n_k}, \frac{Y_k}{\sum_{k=1}^{K-1} Y_k}\right),$$

which generalizes to $w(ab, cd) = w(a,c)w(b,d)$ for all $a, b, d, c \in (0,1)$. Let $a = e^{x_1}$,

$b = e^{x_2}$, $c = e^{x_3}$, $d = e^{x_4}$. Then, we have

$$\varphi(x_1 + x_2, x_3 + x_4) = \varphi(x_1, x_3) + \varphi(x_2, x_4)$$

for all $x_1, x_2, x_3, x_4 \in (-\infty, 0)$ where $\varphi(x, y) = \ln w(e^x, e^y)$. By Theorem 1 on page 215 in Aczél (1966), we get $\varphi(x, y) = d_1 x + d_2 y$ for some constants d_1, d_2 which implies $w(\pi, \theta) = \pi^{d_1}\theta^{d_2}$ for all $\pi, \theta \in (0, 1)$. Now, using weak reflexivity, we get

$$w(t\pi, t\theta) + w\big((1-t)\pi, (1-t)\theta\big) = w(\pi, \theta)$$

for any $t \in (0, 1)$, which simplifies to $t^{d_1+d_2} + (1-t)^{d_1+d_2} = 1$. Since the left-hand side is strictly increasing in $d_1 + d_2$, then $d_1 + d_2 = 1$ is the unique solution.

Thus, we have

$$W_K\big((I_1, n_1, Y_1), \ldots, (I_K, n_K, Y_K)\big) = f^{-1}\left(\sum_{k=1}^{K} \pi_k^{1-\alpha}\theta_k^{\alpha} f(I_k)\right)$$

for all $K \geq 2$, $(I_k)_{k=1}^K \in \mathbb{R}_+^K$, $(n_k, Y_k)_{k=1}^K \in \mathbb{R}_{++}^{2K}$, and for some $\alpha \in \mathbb{R}$. Due to continuity of W_K, these results extend to cases where $n_k = 0$ or $Y_k = 0$ for some k.

A.2 Proof of Proposition 1

In the proof, we use Proposition 8 and the following lemma.

Lemma 1. *There exist two distinct distributions that share the same moments and Gini coefficients, which are all finite.*

Proof. Let X_c be a random variable with the following density function

$$f_c(x) = \big(1 + c\sin(2\pi\ln(x))\big)\frac{1}{\sqrt{2\pi}}\mathbb{1}_{[0,\infty)}(x)\frac{1}{x}e^{-\frac{(\ln(x))^2}{2}}$$

that depends on parameter $c \in \mathbb{R}$. Note that X_0 is distributed lognormally with parameters $\mu = 0$, $\sigma = 1$. It can be shown that for $c \in [-1, 1]$, all moments of X_c are finite and do not depend on c.[27] Moreover, we show that there are two distinct values of c such that the Gini coefficients of X_c are equal. Since the Gini coefficient can be written as $G = 1 - 2\int_0^1 L(x)/L(1)dx$, it suffices to show that $\int_0^1 L_c(x)dx$ is equal for two different values of c, where L_c is the generalized Lorenz curve of X_c.

[27] See, for example, Schmüdgen (2017).

Since $L_c(F_c(x)) = \int_0^x tf_c(t)dt$, where F_c is the CDF of X_c, we need to show that there exist two distinct values of c for which the integral of the generalized Lorenz curve, $A_c = \int_0^1 \int_0^{F_c^{-1}(x)} tf_c(t)dtdx$, are equal. With some manipulation, we get

$$A_c = \frac{1}{\sqrt{2\pi}}(a + bc + dc^2),$$

where

$$\begin{aligned}
a &= \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} e^z \left(1 - \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds\right) dz \\
b &= \int_{-\infty}^{\infty} \sin(2\pi z) e^{-\frac{z^2}{2}} e^z dz - \int_{-\infty}^{\infty} \sin(2\pi z) e^{-\frac{z^2}{2}} e^z \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} dsdz \\
&\quad - \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} e^z \int_{-\infty}^{z} \sin(2\pi s) \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} dsdz \\
d &= \int_{-\infty}^{\infty} \sin(2\pi z) e^{-\frac{z^2}{2}} e^z \int_{-\infty}^{z} \sin(2\pi s) \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} dsdz.
\end{aligned}$$

Now, if $b = 0$, then we have the result since $A_c = A_{-c}$ for any $c \in [-1, 1]$. First, note that

$$\int_{-\infty}^{\infty} \sin(2\pi z) e^{-\frac{z^2}{2}} e^z dz = \int_{-\infty}^{\infty} \sin(2\pi z) e^{-\frac{1}{2}(z-1)^2} e^{\frac{1}{2}} dz = e^{\frac{1}{2}} \int_{-\infty}^{\infty} \sin(2\pi x) e^{-\frac{1}{2}(x)^2} dx = 0$$

as an integral of an odd function. Thus,

$$\begin{aligned}
b &= -\int_{-\infty}^{\infty} \sin(2\pi z) e^{-\frac{z^2}{2}} e^z \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} dsdz - \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} e^z \int_{-\infty}^{z} \sin(2\pi s) \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} dsdz \\
&= -e^{\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{z} e^{-\frac{1}{2}(z-1)^2} e^{-\frac{s^2}{2}} \big(\sin(2\pi z) + \sin(2\pi s)\big) dsdz \\
&= -2e^{\frac{1}{2}} \int_{-\infty}^{1} e^{-\frac{1}{4}t^2} \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}t)^2} \big(\sin(2\pi(x + \tfrac{1}{2}t)) \cos(-\pi t)\big) dxdt \\
&= -2e^{\frac{1}{2}} \int_{-\infty}^{1} e^{-\frac{1}{4}t^2} \cos(-\pi t) \int_{-\infty}^{\infty} e^{-s^2} \sin(2\pi s) dsdt = 0,
\end{aligned}$$

since $\int_{-\infty}^{\infty} e^{-s^2} \sin(2\pi s)ds = 0$ as an integral of an odd function. $\square$

Suppose there exists a finite set of moments Ω and a function F such that (4) holds. Suppose L_1 and L_2 are generalized Lorenz curves of two distinct distributions that have the same values for the moments in Ω and the same Gini coefficients. Consider a population partitioned into K subgroups where each subgroup has generalized Lorenz curve L_1 and denote the vector of moments of L_1 by Ω_1. Then, the aggregate Gini coefficient of the population is equal to $G = F\big(G(L_1), \ldots, G(L_1), \Omega_1, \ldots, \Omega_1; \pi_1, \ldots, \pi_K\big)$. By Proposition 8, we have

$$F\big(G(L_1), \ldots, G(L_1), \Omega_1, \ldots, \Omega_1; \pi_1, \ldots, \pi_K\big) = \Big(\sum_{k=1}^{K} \sqrt{\pi_k \theta_k G(L_1)}\Big)^2 = G(L_1),$$

since subgroup Lorenz curves are homothetic, $\theta_k = \pi_k$ for all k, and $\sum_{k=1}^{K} \pi_k = 1$.

Next, consider another population partitioned into K subgroups, where one subgroup has generalized Lorenz curve L_1 and the other subgroups have generalized Lorenz curves L_2, and let $\tilde{G}$ denote the aggregate Gini coefficient of this population. Since the subgroup Gini coefficients and moments are the same as before, we have

$$\tilde{G} = F\big(G(L_1), \ldots, G(L_1), \Omega_1, \ldots, \Omega_1; \pi_1, \ldots, \pi_K\big) = G(L_1).$$

But since the subgroup Lorenz curves are not all homothetic, then Proposition 8 implies $\tilde{G} > G(L_1)$, which is a contradiction. Therefore, (4) cannot hold.

Note that, as is shown in the proof of Lemma 1, there exist two distinct distributions that share the same sequence of moments and Gini coefficients. Therefore, equation (4) would not hold even if we allowed Ω to be a countably infinite set of moments.

A.3 Proof of Theorem 2

We first show that the within-group inequality term (2) and between-group inequality term (3) sum up to the aggregate Gini coefficient by showing that $G^B = G - G^W$. Let $\Delta = \mathbb{E}\big[|y_i - y_j|\big]$ denote the mean absolute difference in the population. Note that the aggregate Gini coefficient is equal to $\Delta/2\mu$ and the within-group inequality term $G^W = \big(\sum_{k=1}^{K} \pi_k \sqrt{\Delta_k}\big)^2/2\mu$. Define $\Theta = \mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j|\big]$. Now,

$$\begin{aligned}
G - G^W &= \frac{1}{2\mu}\Delta - \frac{1}{2\mu}\Bigg(\sum_{k=1}^{K} \pi_k \sqrt{\Delta_k}\Bigg)^2 = \frac{1}{2\mu}\Bigg(\Delta - \sum_{k=1}^{K} \pi_k^2 \Delta_k - \sum_{k \neq l} \pi_k \pi_l \sqrt{\Delta_k \Delta_l}\Bigg) \\
&= \frac{1}{2\mu}\Bigg(\mathbb{E}\big[|y_i - y_j|\big] - \sum_{k=1}^{K} \mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j|\big] - \sum_{k \neq l} \pi_k \pi_l \sqrt{\Delta_k \Delta_l}\Bigg) \\
&= \frac{1}{2\mu}\Bigg(\Theta - \sum_{k \neq l} \pi_k \pi_l \sqrt{\Delta_k \Delta_l}\Bigg) = G^B.
\end{aligned}$$

Next, we show that the decomposition satisfies all the axioms. Since the within-group inequality term has the form of (1) with $\alpha = \frac{1}{2}$ and $f(x) = x^{\frac{1}{2}}$, it satisfies axioms 1-6 by Theorem 1. If all subgroups have identical income distributions, then the Gini coefficients are equal across subgroups and the income share of each subgroup equals its population share. In this case, both the aggregate Gini coefficient and the within-group inequality term in (2) are equal to the common subgroup Gini coefficient. Thus, between-group inequality is equal to zero and Axiom 7 holds.

Bhattacharya and Mahalanobis (1967) show that the aggregate Gini coefficient can be written as

$$G(L) = \sum_k \pi_k \theta_k G_k + G(\bar{L}) + R, \tag{10}$$

where $\bar{L}$ is the generalized Lorenz curve after replacing each individual's income with the respective subgroup mean and R is a residual term that depends on the amount of overlap between the subgroup income distributions. R is zero if and only if subgroup income distributions do not overlap. It is possible to create overlap with within-group transfers that keep subgroup Gini coefficients constant whenever at least two subgroups have positive population and income share. Hence, the aggregate Gini coefficient is aggregative if and only if at most one subgroup, j, has both strictly positive population and income share. In this special case, the between-group inequality term (3) reduces to $G^B = \pi_j + \theta_j - \pi_j \theta_j$, which does not depend on subgroup income distributions under fixed aggregate characteristics. Hence, Axiom 8 holds.

We next show that the decomposition of Gini coefficient into (2) and (3) is the unique decomposition that satisfies axioms 1-8. Suppose W is a within-group inequality term for the Gini coefficient that satisfies all eight axioms. First, by Theorem 1,

$$W\big((G_1, n_1, Y_1), \ldots, (G_K, n_K, Y_K)\big) = f^{-1}\big(\sum_{k=1}^{K} \pi_k^{1-\alpha} \theta_k^{\alpha} f(G_k)\big),$$

for all $(G_k, n_k, Y_k)_{k=1}^{K} \in \mathbb{R}_+^{3K}$, where f is a strictly monotonic function with $f(0) = 0$ and $\alpha \in \mathbb{R}$. Since cf generates the same W as f for any $c \neq 0$, we can assume $f(1) = 1$ without loss of generality.

Now, if there is only one subgroup, j, with strictly positive population and income share, then by Axiom 8 and equation (10), we get

$$W = f^{-1}\big(\pi_j^{1-\alpha} \theta_j^{\alpha} f(G_j)\big) = \pi_j \theta_j G_j$$

for all $\pi_j, \theta_j, G_j \in [0,1]$. By taking f of both sides and setting $G_j = 1$, we get $\pi_j^{1-\alpha}\theta_j^{\alpha} = f(\pi_j \theta_j)$ for all $\pi_j, \theta_j \in [0,1]$. By setting $\pi_j = 1$, we get $f(x) = x^{\alpha}$ for all $x \in [0,1]$, and by setting $\theta_j = 1$, we get $f(x) = x^{1-\alpha}$ for all $x \in [0,1]$. Thus, we have $\alpha = 1-\alpha$ which implies $\alpha = \frac{1}{2}$ and $f(x) = x^{\frac{1}{2}}$ for all $x \in [0,1]$. Therefore, the decomposition with the within-group inequality term (2) and between-group inequality term (3) is the unique decomposition for the Gini coefficient that satisfies axioms 1-8.

A.4 Proof of Proposition 2

In the proof, we use the following lemma.

Lemma 2. *Merging the first $m \leq K$ subgroups into one weakly increases the within-group inequality of the Gini coefficient. That is,*

$$\begin{aligned}
&G^W\big((G_1, \pi_1, \theta_1), \ldots, (G_K, \pi_K, \theta_K)\big) \\
&\leq G^W\big((\tilde{G}, \tilde{\pi}, \tilde{\theta}), (G_{m+1}, \pi_{m+1}, \theta_{m+1}), \ldots, (G_K, \pi_K, \theta_K)\big),
\end{aligned}$$

where $\tilde{\pi} = \sum_{k=1}^{m} \pi_k$, $\tilde{\theta} = \sum_{k=1}^{m} \theta_k$, and $\tilde{G}$ is the Gini coefficient of the mixture of the first m subgroups' income distributions with weights $(\pi_k/\tilde{\pi})_{k=1}^{m}$. Merging does not affect within-group inequality if and only if the income distributions of the m subgroups are identical.

Proof. Let G_m^W denote the within-group inequality among the first m subgroups and let $\tilde{G}^W$ denote the level of within-group inequality after the first m subgroups are merged into one. Because $\tilde{G} \geq G_m^W = \big(\sum_{k=1}^{m} \sqrt{(\pi_k/\tilde{\pi})(\theta_k/\tilde{\theta})G_k}\big)^2$, we have that

$$\begin{aligned}
\tilde{G}^W &= \Big(\sqrt{\tilde{\pi}\tilde{\theta}\tilde{G}} + \sum_{k=m+1}^{K} \sqrt{\pi_k \theta_k G_k}\Big)^2 \\
&\geq \Bigg(\sqrt{\tilde{\pi}\tilde{\theta}} \sum_{k=1}^{m} \sqrt{\frac{\pi_k}{\tilde{\pi}} \frac{\theta_k}{\tilde{\theta}} G_k} + \sum_{k=m+1}^{K} \sqrt{\pi_k \theta_k G_k}\Bigg)^2 = \Big(\sum_{k=1}^{K} \sqrt{\pi_k \theta_k G_k}\Big)^2 = G^W.
\end{aligned}$$

Moreover, we have $\tilde{G} = G_m^W$ if and only if the income distributions of the first m subgroups are identical, in which case we have $\tilde{G}^W = G^W$. □

The proof proceeds by showing that within-group inequality increases under the operation while the aggregate Gini coefficient is unaffected. Consider first drawing a random sample of fraction $\alpha > 0$ from the population and assigning the sampled individuals to a new subgroup. This operation increases within-group inequality term. To show this, we use equation (2) and the Brunn-Minkowski theorem (Proposition 8),

$$\begin{aligned}
&G^W\big((G_1,(1-\alpha)\pi_1,(1-\alpha)\theta_1),\ldots,(G_K,(1-\alpha)\pi_K,(1-\alpha)\theta_K),(G,\alpha,\alpha)\big)\\
&=\Big((1-\alpha)\sum_{k=1}^{K}\sqrt{\pi_k\theta_k G_k}+\alpha\sqrt{G}\Big)^2 \geq \Big((1-\alpha)\sum_{k=1}^{K}\sqrt{\pi_k\theta_k G_k}+\alpha\sum_{k=1}^{K}\sqrt{\pi_k\theta_k G_k}\Big)^2\\
&=\Big(\sum_{k=1}^{K}\sqrt{\pi_k\theta_k G_k}\Big)^2 = G^W\big((G_1,\pi_1,\theta_1),\ldots,(G_K,\pi_K,\theta_K)\big),
\end{aligned} \tag{11}$$

where equality holds if and only if the subgroups have identical income distributions.

Now, by the weak reflexivity, symmetry, and replacement axioms, we have

$$\begin{aligned}
&G^W\big((G_1,(1-\alpha)\pi_1,(1-\alpha)\theta_1),\ldots,(G_K,(1-\alpha)\pi_K,(1-\alpha)\theta_K),(G,\alpha,\alpha)\big)\\
&=G^W\big((G_1,(1-\alpha)\pi_1,(1-\alpha)\theta_1),\ldots,(G_K,(1-\alpha)\pi_K,(1-\alpha)\theta_K),\\
&\qquad (G,\alpha\pi_1,\alpha\pi_1),\ldots,(G,\alpha\pi_K,\alpha\pi_K)\big).
\end{aligned} \tag{12}$$

Let $\tilde{G}_k$ and $\tilde{\theta}_k$ denote the resulting Gini coefficient and income share after a $1-\alpha$ sample of subgroup k's distribution is merged with a $\alpha\pi_k$ sample of the aggregate distribution. By Lemma 2 and symmetry, the right-hand side of (12) is less than or equal to $G^W\big((\tilde{G}_1,\pi_1,\tilde{\theta}_1),\ldots,(\tilde{G}_K,\pi_K,\tilde{\theta}_K)\big)$. Equality holds only if all subgroup have the same distribution of income. Combining this with (11) we get

$$G^W\big((\tilde{G}_1,\pi_1,\tilde{\theta}_1),\ldots,(\tilde{G}_K,\pi_K,\tilde{\theta}_K)\big) \geq G^W\big((G_1,\pi_1,\theta_1),\ldots,(G_K,\pi_K,\theta_K)\big)$$

The left-hand side of above equation corresponds to within-group inequality after replacing the incomes of a random α sample with a random draw from the aggregate income distribution. As the aggregate Gini coefficient is unchanged while the within-group inequality term increases, between-group inequality must decrease. Within-group inequality remains unchanged if and only if the subgroup income distributions are identical. In this case, between-group inequality is also unaffected.

A.5 Proof of Proposition 3

By Lemma 2 and symmetry, merging any $m \leq K$ subgroups weakly increases within-group inequality. Because the aggregate Gini coefficient is unaffected by merging subgroups, between-group inequality must weakly decrease. Moreover, by Lemma 2, within-group inequality does not change if and only if the m subgroups have identical income distributions. Then between-group inequality does not change either.

A.6 Proof of Proposition 4

Let $\tilde{G}$ denote the Gini coefficient after replacing y_i with $\tilde{y}_i$ for all i. First, note that the redistribution scheme reduces the aggregate Gini coefficient by α percent:

$$\tilde{G} = \frac{\mathbb{E}\big[|\tilde{y}_i - \tilde{y}_j|\big]}{2\mu} = \frac{\mathbb{E}\Big[|y_i - \alpha(y_i - \mu) - \big(y_j - \alpha(y_j - \mu)\big)|\Big]}{2\mu} = (1-\alpha)\frac{\mathbb{E}\big[|y_i - y_j|\big]}{2\mu}$$
$$= (1-\alpha)G.$$

It is easy to see that within-group inequality also decreases by α percent:

$$\begin{aligned}
\tilde{G}^W &= \left(\sum_k \sqrt{\pi_k \tilde{\theta}_k \tilde{G}_k}\right)^2 = \left(\sum_k \sqrt{\pi_k \tilde{\theta}_k \frac{\mathbb{E}_k\big[|\tilde{y}_i - \tilde{y}_j|\big]}{2\tilde{\mu}_k}}\right)^2 \\
&= (1-\alpha)\left(\sum_k \sqrt{\pi_k \pi_k \frac{\mathbb{E}_k\big[|y_i - y_j|\big]}{2\mu}}\right)^2 = (1-\alpha)\left(\sum_k \sqrt{\pi_k \theta_k \frac{\mathbb{E}_k\big[|y_i - y_j|\big]}{2\mu_k}}\right)^2 \\
&= (1-\alpha)\left(\sum_k \sqrt{\pi_k \theta_k G_k}\right)^2 = (1-\alpha)G^W,
\end{aligned}$$

where $\tilde{\theta}_k$ is subgroup k's income share after the replacement of incomes. Finally, since both the aggregate Gini coefficient as well as the within-group term decrease by fraction α, between-group term must also decrease by fraction α.

A.7 Proof of Proposition 5

Note that the within-group inequality term can be written as

$$G^W = \left(\sum_k \sqrt{\pi_k \theta_k G_k}\right)^2 = \left(\sum_k \sqrt{\pi_k \theta_k \frac{\mathbb{E}_k\big[|y_i - y_j|\big]}{2\mu_k}}\right)^2 = \left(\sum_k \pi_k \sqrt{\frac{\mathbb{E}_k\big[|y_i - y_j|\big]}{2\mu}}\right)^2.$$

But since lump-sum transfers between subgroups affect neither the mean absolute difference within subgroups nor the mean income in the population, it follows that such transfers do not affect within-group inequality.

A.8 Proof of Proposition 6

Scale invariance follows directly from the scale invariance of within-group inequality and the inequality measure itself. To show translation invariance, note first that adding some fixed amount z to each income in a population with average income μ decreases the Gini coefficient by a factor of $\mu/(\mu+z)$. Let $\tilde{G}_k$ and $\tilde{\theta}_k$ denote the Gini coefficient and the income share of subgroup k after translation. Within-group inequality after translation can then be computed as

$$\begin{aligned}\tilde{G}^W &= \Big(\sum_k \sqrt{\pi_k \tilde{\theta}_k \tilde{G}_k}\Big)^2 = \Big(\sum_k \sqrt{\pi_k \Big(\pi_k \frac{\mu_k+z}{\mu+z}\Big)\Big(\frac{\mu_k}{\mu_k+z} G_k\Big)}\Big)^2 \\ &= \frac{1}{\mu+z}\Big(\sum_k \sqrt{\pi_k^2 \mu_k G_k}\Big)^2 = \frac{\mu}{\mu+z}\Big(\sum_k \sqrt{\pi_k \theta_k G_k}\Big)^2 = \frac{\mu}{\mu+z} G^W,\end{aligned}$$

that is, within-group inequality also decreases by a same factor of $\mu/(\mu+z)$. Hence, the ratio of within-group inequality to between-group inequality remains unchanged.

A.9 Proof of Proposition 8

The inequality together with the equality condition follows directly from the Brunn-Minkowski theorem. See for example Gardner (2002) for the proof of the Brunn-Minkowski theorem in the case of two sets. The theorem can be easily generalized to three sets. The general case of K sets follows by induction.

By associativity of Minkowski addition, we have

$$\begin{aligned}|\lambda_1+\lambda_2+\lambda_3| &= |(\lambda_1+\lambda_2)+\lambda_3| \geq \Big(|\lambda_1+\lambda_2|^{\frac{1}{2}} + |\lambda_3|^{\frac{1}{2}}\Big)^2 \\ &\geq \Big(\big((|\lambda_1|^{\frac{1}{2}} + |\lambda_2|^{\frac{1}{2}})^2\big)^{\frac{1}{2}} + |\lambda_3|^{\frac{1}{2}}\Big)^2 = \Big(|\lambda_1|^{\frac{1}{2}} + |\lambda_2|^{\frac{1}{2}} + |\lambda_3|^{\frac{1}{2}}\Big)^2,\end{aligned}$$

where the inequalities hold as equalities when λ_1, λ_2, and λ_3 are homothetic. The last claim follows from the fact that homothety is closed under Minkowski addition.

A.10 Proof of Proposition 9

Transfers that increase subgroup inequality by the Lorenz criterion shift the subgroup Lorenz curves outwards so that the pre-transfers Lorenz regions are subsets of the post-transfers Lorenz regions. Since $A \oplus B \subseteq \tilde{A} \oplus \tilde{B}$ if $A \subseteq \tilde{A}$ and $B \subseteq \tilde{B}$, such transfers also shift the aggregate Lorenz curve outwards. Hence, aggregate inequality must increase.

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Online Appendix

A Extension of the Axiomatic Framework to Other Inequality Indices

In the main paper, we introduce an axiomatic framework to define decomposability of inequality indices and show that there is a unique subgroup decomposition for the Gini coefficient. In this appendix, we discuss how our framework relates to the existing definitions of decomposability and show that our framework can also be used to derive the decompositions for the generalized entropy indices and the Foster-Shneyerov indices, two families of inequality indices that are commonly considered decomposable although under different frameworks.

A.1 Definitions of Decomposability

In a series of seminal papers, Bourguignon (1979), Shorrocks (1980, 1984), Cowell (1980), and Cowell and Kuga (1981) define decomposability in terms of a strict *aggregativity* requirement. In their definition, an inequality index is decomposable if aggregate inequality is a function of subgroup inequality indices, average incomes, and population shares. The main result from this literature is that any inequality measure which satisfies this aggregativity requirement is ordinally equivalent to a generalized entropy index. These indices include the Theil index, the mean log deviation, and half the squared coefficient of variation, but not the Gini coefficient.

Foster and Shneyerov (2000) explore a different requirement for decomposability which they call *path independence*. They define within-group inequality and between-group inequality in terms of representative incomes. Within-group inequality is the level of inequality when all subgroup distributions are re-scaled to have a common representative income level. Between-group inequality is the level of inequality when each individual's income is replaced with their subgroup's representative income. An inequality index is decomposable if it can be written as the sum of such a within and between-group inequality term. From this definition, Foster and Shneyerov (2000) derive a family of decomposable inequality indices that is uniquely determined (up to a scalar multiple) by their path independence requirement. This family includes the mean log deviation and half the variance of logs, but not the Gini coefficient.[1]

Interestingly, the Foster-Shneyerov indices do not satisfy the aggregativity requirement by Bourguignon (1979) and others, while the generalized entropy indices do not satisfy the path independence requirement by Foster and Shneyerov (2000). Both these families of inequality indices are nevertheless commonly considered decomposable. In this appendix, we show that the standard decomposition formulas for the generalized entropy and the Foster-Shneyerov indices are consistent with the axiomatic framework introduced in section III of the this paper. Moreover, we show that if we strengthen Axiom 7, the decomposition formulas for the generalized entropy and Foster-Shneyerov indices are the unique formulas that satisfy all the axioms. Importantly, strengthening the statement of Axiom 7 does not affect the results for the Gini coefficient. Hence, there exists a common axiomatic framework under which the Gini coefficient, generalized entropy indices, and the Foster-Shneyerov indices are all decomposable with unique decomposition formulas.

We also show that for Atkinson indices, which satisfy aggregativity, there does not exist a decomposition that satisfies our axioms. The fact that the Gini coefficient is decomposable under our framework while the Atkinson indices are not shows that aggregativity is neither a necessary nor sufficient condition for decomposability.

A.2 Definitions

The inequality indices considered in this section require a more restricted domain compared to the Gini coefficient as they are only defined distributions for which a cumulative distribution function exists. Thus, in this appendix, we represent distributions with their cumulative distribution functions. Generalized entropy indices are a family of inequality indices defined as

$$GE_\alpha(F) = \begin{cases} \frac{1}{\alpha(\alpha-1)} \int \left(\left(\frac{y}{\mu}\right)^\alpha - 1 \right) dF(y) & \text{for } \alpha \neq 0, 1 \\ \int \ln \frac{\mu}{y} dF(y) & \text{for } \alpha = 0 \\ \int \frac{y}{\mu} \ln \frac{y}{\mu} dF(y) & \text{for } \alpha = 1, \end{cases} \tag{13}$$

where μ is the mean income, $\alpha \in \mathbb{R}$ is a parameter that indexes different members of the family, and F is the cumulative distribution function. The Theil index is the generalized entropy index with $\alpha = 1$, and the mean log deviation is the generalized

[1]Note that the variance of logs is not, strictly speaking, an inequality measure as it violates the Pigou-Dalton principle, see Foster and Ok (1999).

entropy index with $\alpha = 0$.

The Foster-Shneyerov family, introduced in Foster and Shneyerov (2000), is defined as

$$FS_q(F) = \begin{cases} \frac{1}{q} \ln \frac{\mu_q(F)}{g(F)} & \text{for } q \neq 0 \\ \frac{1}{2}\sigma^2_{\ln} & \text{for } q = 0, \end{cases} \tag{14}$$

where $\mu_q(F) = \left(\int y^q dF(y)\right)^{\frac{1}{q}}$ is the power mean of order q, $g(F) = \exp(\int \ln y dF(y))$ is the geometric mean, and $\sigma^2_{\ln} = \int (\ln y - \ln \mu)^2 dF(y)$ is the variance of log income.[2]

Finally, the family of Atkinson indices, introduced in Atkinson (1970), is defined as

$$A_\varepsilon(F) = 1 - \frac{\mu_{1-\varepsilon}(F)}{\mu} \qquad \text{for } \varepsilon < 1,$$

where $\mu_{1-\varepsilon}(F) = \left(\int y^{1-\varepsilon} dF(y)\right)^{\frac{1}{1-\varepsilon}}$ is the power mean of order $1 - \varepsilon$.

A.3 Aggregativity and Axiom 7

In the main text, Axiom 7 states that between-group inequality is equal to zero when all subgroup income distributions are identical. As shown in Proposition 1, aggregate Gini coefficient in general depends on all moments of the subgroup income distributions. Hence, we want to allow between-group inequality to be positive when subgroups differ in any moments.

However, it is reasonable to strengthen Axiom 7 for the generalized entropy and Foster-Shneyerov indices. The generalized entropy indices depend only on subgroup means, inequality levels, and population shares, and it is therefore natural that between-group inequality is zero whenever the subgroups all have the same mean. Similarly, the Foster-Shneyerov index of order q, conditional on subgroup inequality levels and population shares, depends only on subgroup moments of order q, and it is therefore natural that between-group inequality is zero whenever the subgroup income distributions have the same qth moment. Hence, the statement of Axiom 7 should reflect the aggregativity properties of the underlying inequality index. For this purpose, we define the concept of Ω-aggregativity.

Definition 1. *An inequality measure $I \colon \mathcal{D} \to \mathbb{R}_+$ is Ω-aggregative if there exists a finite*

[2] Note that, strictly speaking, only Foster-Shneyerov indices with parameter values $q \geq 1$ are inequality indices. For other parameter values, the Pigou-Dalton transfer principle is violated.

set of moments Ω *and a function* $F\colon \mathbb{R}_+^K \times \mathbb{R}_+^{MK} \times \mathbb{R}_+^K \to \mathbb{R}_+$ *such that*

$$I = F\big(I_1, \ldots, I_K, \Omega_1, \ldots, \Omega_K; n_1, \ldots, n_K\big),$$

where I_k *is the level of inequality in subgroup* k*,* Ω_k *are subgroup* k*'s values for the moments contained in* Ω*,* Ω *is the minimal set of moments such that the above formula holds, and* M *is the cardinality of* Ω*. If such an* Ω *and* F *do not exist, then* I *is* Ω*-nonaggregative.*

Generalized entropy indices and Atkinson indices are inequality measures that can be written as functions of subgroup inequality levels, average incomes, and population sizes. Hence, these measures are Ω-aggregative with $\Omega = \{\mu\}$. We call inequality indices with this property means-aggregative. The Foster-Shneyerov index of order $q \neq 0$ is aggregative with qth moments, that is $\Omega = \{\mu_q\}$.[3] In Proposition 1 in section IV, we show that the aggregate Gini coefficient can generally not be computed from subgroup Gini coefficients, population sizes, and any number of subgroup moments. Thus, the Gini coefficient is Ω-nonaggregative. We use Ω-aggregativity to introduce an equivalence relation between distributions, which we call Ω-equivalence.

Definition 2. *For a given inequality index, we call two distributions* Ω*-equivalent if they are identical in all moments contained in* Ω*. If* Ω *does not exist, then we call two distributions* Ω*-equivalent if they are the same distribution.*

With Ω-equivalence, Axiom 7 can be strengthened as follows:

7*. **Normalization.** B_K is equal to zero if and only if all subgroup income distributions are Ω-equivalent.

Note that the Gini coefficient satisfies the statement of Axiom 7*, which is a direct corollary of Axiom 8 and the fact the Gini coefficient is Ω-nonaggregative. Hence, the Gini coefficient is decomposable with the same unique decomposition formula even under this strengthened version of Axiom 7.

A.4 Decompositions of the Generalized Entropy, Foster-Shneyerov, and Atkinson Indices

We next show axioms 1–6, 7*, and 8, uniquely determine the standard decomposition formulas for the generalized entropy indices and the Foster-Shneyerov indices.

[3] Note also that it is possible to construct an Ω-aggregative inequality index for any arbitrary choice of Ω by taking convex combinations of appropriate Foster-Shneyerov indices.

Proposition 10. *For all $\alpha \in \mathbb{R}$, the decomposition of generalized entropy index* (13)

$$GE_\alpha(F) = \sum_{k=1}^{K} \pi_k^{1-\alpha} \theta_k^\alpha GE_\alpha(F_k) + GE_\alpha(\bar{F}), \tag{15}$$

where $\bar{F}$ is the cumulative distribution function after replacing each individual's income with the respective subgroup mean, is the unique decomposition that satisfies axioms 1–6, 7^*, and 8.

Proof. The within-group inequality term in (15) is an arithmetic average of subgroup inequalities with a geometric average weight structure in the subgroups' income and population shares. It therefore satisfies axioms 1 through 6 as a result of Theorem 1. Because generalized entropy indices are means-aggregative, Axiom 7^* requires that between-group inequality is zero if and only if all subgroups have equal means. The between-group inequality term in (15) is clearly equal to zero if and only if all subgroups have equal means. Moreover, the between-group inequality term is always constant for given aggregate characteristics and therefore satisfies Axiom 8.

Next, we show that (15) is the unique decomposition for generalized entropy indices that satisfies axioms 1-8. Let $\alpha \in \mathbb{R}$ and suppose that $GE_\alpha = W + B$ is a decomposition of a generalized entropy index of parameter α that satisfies axioms 1-8. Then, by Axiom 8, B is independent of subgroup income distributions. Hence, we can eliminate inequality within each subgroup without changing the value of B by replacing each individual's income by their subgroup's average income. By Axiom 4, $W(\bar{F}) = 0$, and thus $B = GE_\alpha(\bar{F})$. Finally, we get $W = \sum_{k=1}^{K} \pi_k^{1-\alpha} \theta_k^\alpha GE_\alpha(F_k)$ by subtracting B from $GE_\alpha(F)$.

□

Proposition 11. *For all $q \in \mathbb{R}$, the decomposition of Foster-Shneyerov index* (14)

$$FS_q(F) = \sum_{k=1}^{K} \pi_k FS_q(F_k) + FS_q(\tilde{F}), \tag{16}$$

where $\tilde{F}$ is the cumulative distribution function after replacing each individual's income with the respective subgroup power mean of order q, is the unique decomposition that satisfies axioms 1–6, 7^*, and 8.

Proof. Let $q \in \mathbb{R}$. Since equation (16) always holds, the Foster-Shneyerov index with parameter q is Ω-aggregative with Ω containing the qth order moment.

The within-group inequality term in equation (16) fits the form of (1) with $f(x) = x$ and $\alpha = 0$. Thus, by Theorem 1, (16) satisfies axioms 1-6. The inequality of power means[4] states that power means with different exponents give the same value if and only if all arguments are equal. Thus, between-group inequality in (16) is zero if and only if all subgroups have equal qth order moments, that is, if and only if subgroup income distributions are Ω-equivalent. Hence, Axiom 7* holds. Finally, the condition in Axiom 8 never holds for any Foster-Shneyerov index with $q \neq 1$, and in the case of $q = 1$, the index and the decomposition coincides with the generalized entropy index of parameter $\alpha = 0$, which is shown to satisfy all axioms in Proposition 10. Thus, Axiom 8 holds.

For uniqueness, let $FS_q = W + B$ be a decomposition that satisfies axioms 1-8. Now, consider a case where all subgroups are Ω-equivalent, that is, all subgroups have equal qth moments. By Axiom 7*, between-group inequality is equal to zero and we get $W = FS_q(F)$. Using equation (16), we get

$$FS_q(F) = \sum_k \pi_k FS_q(F_k) = W$$

for all $\pi_k \in [0, 1]$ and $FS_q(F_k) \in \mathbb{R}_+$.

Using Theorem 1, we get

$$f^{-1}\Big(\sum_{k=1}^{K} \pi_k^{1-\alpha}\theta_k^{\alpha} f\big(FS_q(F_k)\big)\Big) = \sum_k \pi_k FS_q(F_k)$$

or

$$\sum_k \pi_k^{1-\alpha}\theta_k^{\alpha} f\big(FS_q(F_k)\big) = f\Big(\sum_k \pi_k FS_q(F_k)\Big) \tag{17}$$

for all $\pi_k \in [0, 1]$, $\theta_k \in [0, 1]$, and $FS_q(F_k) \in \mathbb{R}_+$, where f is a strictly monotonic function with $f(0) = 0$ and $\alpha \in \mathbb{R}$. Since cf gives the same W as f for any $c \neq 0$, we can assume $f(1) = 1$ without loss of generality. By setting $FS_q(F_k) = 1$ for some k and $FS_q(F_j) = 0$ for all $j \neq k$, we get $\pi_k^{1-\alpha}\theta_k^{\alpha} = f(\pi_k)$ for all $\pi_k, \theta_k \in [0, 1]$. Thus, we must have $\alpha = 0$ and hence $f(x) = x$ for all $x \in [0, 1]$. Now, equation (17) becomes

$$\sum_k \pi_k f(FS_q(F_k)) = f\Big(\sum_k \pi_k FS_q(F_k)\Big). \tag{18}$$

for all $\pi_k \in [0, 1]$ and $FS_q(F_k) \in \mathbb{R}_+$. The only strictly monotonic solution to equation

[4] see e.g. Theorem 1 on p. 203 in Bullen (2013)

(18) is $f(x) = ax$ for some $a \neq 0$.[5] With normalization $f(1) = 1$ we get $a = 1$, and thus $W = \sum_k \pi_k FS_q(F_k)$ for all $\pi_k \in [0,1]$, $\theta_k \in [0,1]$, and $FS_q(F_k) \in \mathbb{R}_+$, and we get $B = FS_q(\tilde{F})$ as a residual.

□

We next turn to the Atkinson indices. These indices are aggregative with means and therefore ordinally equivalent to generalized entropy indices (Shorrocks, 1984). However, there is no universally accepted way to decompose Atkinson indices into within-group and between-group inequality terms that sum to aggregate inequality.[6] We show that there cannot exist a decomposition of Atkinson indices that satisfies all of our axioms. In the proof of the proposition, we show that Axioms 4 and 8 imply unique expressions the for within-group and between-group inequality terms, but the implied within-group inequality term does not satisfy Axiom 6 (replacement).

Proposition 12. *For any $\varepsilon < 1$, the Atkinson index A_ε cannot be decomposed into a within-group inequality term and a between-group inequality term that satisfy axioms* $1-6$, 7^*, *and* 8 *and sum to the aggregate Atkinson index.*

Proof. Suppose that $A_\varepsilon = W + B$ is a decomposition of the Atkinson index of parameter ε that satisfies axioms 1-8. Since the Atkinson index for any $\varepsilon < 1$ is a monotonic transformation of a corresponding generalized entropy index, it is means-aggregative. Thus, by Axiom 8, B is always independent of subgroup income distributions. We can thus eliminate inequality within each subgroup by replacing each individual's income by their subgroup's mean income without changing the value of B. Hence, $B = A_\varepsilon(\bar{F})$. By subtracting B from A_ε, we get

$$W = \left(\sum_{k=1}^{K} \pi_k^\varepsilon \theta_k^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}} - \left(\sum_{k=1}^{K} \pi_k^\varepsilon \theta_k^{1-\varepsilon} (1 - A_\varepsilon(F_k))^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}.$$

The above formula for W does not satisfy the replacement axiom, which is a contradiction. Thus, there is no decomposition for Atkinson indices that satisfies axioms 1-8.

□

[5] see Aczél (1966).

[6] The most common decomposition of the Atkinson index was derived by Blackorby et al. (1981) and consists of a within-group inequality term W and a between-group inequality term B that combine into the aggregate Atkinson index as $A = W + B - WB$, and therefore violates our requirement that $A = W + B$.

B Inference

In this section, we derive the asymptotic confidence intervals for the within and between-group inequality terms in the Gini decomposition. We show that the plug-in estimators for the within-group and between-group inequality terms are asymptotically jointly normally distributed, which allows us to derive the asymptotic variance for these terms using the delta method. Furthermore, we confirm through simulation exercises that inference based on the asymptotic variances yields very precise confidence intervals for lognormally distributed data and sample sizes similar to those in our empirical application.

B.1 Asymptotic normality

We consider sampling n i.i.d. random vectors $X_i = (y_i, g_i)$, where $y_i \in \mathbb{R}_+$ is the income and $g_i \in \{1, 2, \ldots, K\}$ is the subgroup affiliation of individual i. Let F denote the cumulative distribution function of y_i, F_k the cumulative distribution function of y_i conditional on $g_i = k$, F_{-k} the cumulative distribution function of y_i conditional on $g_i \neq k$, $\mu = \mathbb{E}[y_i]$, $\mu_k = \mathbb{E}[y_i|g_i = k]$, $\pi_k = \mathbb{P}(g_i = k)$, and $\theta_k = \pi_k\mu_k/\mu$ for $k = 1, \ldots, K$. Moreover, let $\Delta = \mathbb{E}\big[|y_i - y_j|\big]$ denote the mean absolute difference of distribution F, $\Delta_k = \mathbb{E}_k\big[|y_i - y_j|\big|g_i, g_j = k\big]$ the mean absolute difference of subgroup k, $\tilde{\Delta}_k = \mathbb{E}\big[\mathbb{1}_{\{g_i=k\}}\mathbb{1}_{\{g_j=k\}}|y_i - y_j|\big] = \pi_k^2\Delta_k$ the adjusted mean absolute difference of subgroup k, and $\Theta = \mathbb{E}\big[\mathbb{1}_{\{g_i\neq g_j\}}|y_i - y_j|\big]$ the expected absolute difference of incomes between individuals from different subgroups. Finally, let G^W and G^B denote the within-group and between-group inequality terms for the Gini coefficient of distribution F and G_n^W and G_n^B their values computed from a sample of n individuals.

Theorem 3 holds that the pair of within-group and between-group inequality terms of the Gini coefficient is asymptotically jointly normally distributed and states their asymptotic covariance matrix.

Theorem 3. *If $\mathbb{E}[y_i^2]$ exists, $\mu > 0$, and $\tilde{\Delta}_k > 0$ for all k, then*

$$\sqrt{n}\big(G_n^W - G^W, G_n^B - G^B\big) \xrightarrow{d} \mathcal{N}(0, \Sigma),$$

where

$$\Sigma = \begin{bmatrix} \sigma_W^2 & \sigma_{WB} \\ \sigma_{WB} & \sigma_B^2 \end{bmatrix},$$

with the following expressions for the elements in the covariance matrix:

$$\sigma_W^2 = \left(\frac{G^W}{\mu}\right)^2 \mathrm{Var}(y_i) + \sum_{k=1}^{K} \frac{G^W}{2\mu\tilde{\Delta}_k}\sigma_{\tilde{\Delta}_k}^2 - \sum_{k=1}^{K}\left(\frac{G^W}{\mu}\right)^{\frac{3}{2}}\sqrt{\frac{2}{\tilde{\Delta}_k}}\sigma_{\tilde{\Delta}_k,\mu}$$
$$- \sum_{k\neq l} \frac{2G^W}{\mu}\sqrt{\tilde{\Delta}_k\tilde{\Delta}_l} \tag{19}$$
$$\sigma_B^2 = \left(\frac{G^B}{\mu}\right)^2 \mathrm{Var}(y_i) + \frac{1}{4\mu^2}\sigma_\Theta^2 + \sum_{k=1}^{K}\frac{1}{4\mu^2}\left(\frac{\sum_{l\neq k}\sqrt{\tilde{\Delta}_l}}{\sqrt{\tilde{\Delta}_k}}\right)^2\sigma_{\tilde{\Delta}_k}^2 - \frac{G^B}{\mu^2}\sigma_{\Theta,\mu}$$
$$- \sum_{k=1}^{K}\frac{1}{2\mu^2}\frac{\sum_{l\neq k}\sqrt{\tilde{\Delta}_l}}{\sqrt{\tilde{\Delta}_k}}\sigma_{\Theta,\tilde{\Delta}_k} + \sum_{k=1}^{K}\frac{G^B}{\mu^2}\frac{\sum_{l\neq k}\sqrt{\tilde{\Delta}_l}}{\sqrt{\tilde{\Delta}_k}}\sigma_{\mu,\tilde{\Delta}_k}$$
$$- \sum_{k\neq l}\frac{1}{\mu^2}\left(\sum_{i\neq k}\sqrt{\tilde{\Delta}_i}\right)\left(\sum_{j\neq l}\sqrt{\tilde{\Delta}_j}\right)\sqrt{\tilde{\Delta}_k\tilde{\Delta}_l} \tag{20}$$
$$\sigma_{WB} = \frac{1}{2}\left(\sigma_G^2 - \sigma_W^2 - \sigma_B^2\right) \tag{21}$$

where

$$\sigma_G^2 = \left(\frac{G}{\mu}\right)^2 \mathrm{Var}(y_i) + \frac{1}{4\mu^2}\sigma_\Delta^2 - \frac{G}{\mu^2}\sigma_{\Delta,\mu}$$
$$\sigma_\Delta^2 = 4\int_0^\infty\int_0^\infty |x-y|^2 dF(y)dF(x) - 4\Delta^2$$
$$\sigma_{\Delta,\mu} = 2\int_0^\infty\int_0^\infty |x-y|x dF(y)dF(x) - 2\Delta\mu$$
$$\sigma_\Theta^2 = 4\sum_{k=1}^{K}\pi_k(1-\pi_k)^2\int_0^\infty\left(\int_0^\infty |x-y|dF_{-k}(y)\right)^2 dF_k(x) - 4\Theta^2,$$
$$\sigma_{\tilde{\Delta}_k}^2 = 4\pi_k^3\int_0^\infty\left(\int_0^\infty |x-y|dF_k(y)\right)^2 dF_k(x) - 4\tilde{\Delta}_k^2,$$
$$\sigma_{\Theta,\mu} = 2\sum_{k=1}^{K}\pi_k(1-\pi_k)\int_0^\infty\int_0^\infty |x-y|x dF_{-k}(y)dF_k(x) - 2\Theta\mu,$$
$$\sigma_{\tilde{\Delta}_k,\mu} = 2\pi_k^2\int_0^\infty\int_0^\infty |x-y|x dF_k(y)dF_k(x) - 2\tilde{\Delta}_k\mu,$$
$$\sigma_{\Theta,\tilde{\Delta}_k} = 4\pi_k^2(1-\pi_k)\int_0^\infty\int_0^\infty |x-y|dF_{-k}(y)\int_0^\infty |x-y|dF_k(y)dF_k(x) - 4\Theta\tilde{\Delta}_k.$$

Proof. First, note that

$$
\begin{aligned}
G^W &= \left(\sum_{k=1}^K \sqrt{\pi_k\theta_k G_k}\right)^2 = \left(\sum_{k=1}^K \sqrt{\pi_k\theta_k \frac{\mathbb{E}\big[|y_i-y_j|\big|g_i,g_j=k\big]}{2\mu_k}}\right)^2 \\
&= \left(\sum_{k=1}^K \sqrt{\pi_k^2\frac{\mu_k}{\mu} \frac{\mathbb{E}\big[|y_i-y_j|\big|g_i,g_j=k\big]}{2\mu_k}}\right)^2 = \frac{1}{2\mu}\left(\sum_{k=1}^K \pi_k\sqrt{\mathbb{E}\big[|y_i-y_j|\big|g_i,g_j=k\big]}\right)^2 \\
&= \frac{1}{2\mu}\left(\sum_{k=1}^K \pi_k\sqrt{\Delta_k}\right)^2 = \frac{1}{2\mu}\left(\sum_{k=1}^K \sqrt{\tilde{\Delta}_k}\right)^2.
\end{aligned}
$$

That is, we can write the within-group inequality term in the Gini decomposition as a function of the aggregate mean and the subgroups' adjusted mean absolute differences. Similarly,

$$
\begin{aligned}
G^B = G - G^W &= \frac{1}{2\mu}\Delta - \frac{1}{2\mu}\left(\sum_{k=1}^K \sqrt{\tilde{\Delta}_k}\right)^2 = \frac{1}{2\mu}\left(\Delta - \sum_{k=1}^K \tilde{\Delta}_k - \sum_{k\neq l}\sqrt{\tilde{\Delta}_k\tilde{\Delta}_l}\right) \\
&= \frac{1}{2\mu}\left(\mathbb{E}\big[|y_i-y_j|\big] - \sum_{k=1}^K \mathbb{E}\big[\mathbb{1}_{\{g_i=k\}}\mathbb{1}_{\{g_j=k\}}|y_i-y_j|\big] - \sum_{k\neq l}\sqrt{\tilde{\Delta}_k\tilde{\Delta}_l}\right) \\
&= \frac{1}{2\mu}\left(\Theta - \sum_{k\neq l}\sqrt{\tilde{\Delta}_k\tilde{\Delta}_l}\right).
\end{aligned}
$$

Let

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^n y_i,$$

$$\hat{\Theta} = \frac{1}{n(n-1)}\sum_{i\neq j}\mathbb{1}_{\{g_i\neq g_j\}}|y_i-y_j|,$$

and

$$\hat{\tilde{\Delta}}_k = \frac{1}{n(n-1)}\sum_{i\neq j}\mathbb{1}_{\{g_i=k\}}\mathbb{1}_{\{g_j=k\}}|y_i-y_j|.$$

Since $\hat{\mu}$, $\hat{\Theta}$, and $\hat{\tilde{\Delta}}_k$ are U-statistics, then by Theorem 7.1 in Hoeffding (1948) they are asymptotically jointly normally distributed with covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_\mu^2 & \sigma_{\mu\Theta} & \sigma_{\mu\tilde{\Delta}_1} & \cdots & \sigma_{\mu\tilde{\Delta}_K} \\ \sigma_{\Theta\mu} & \sigma_\Theta^2 & \sigma_{\Theta\tilde{\Delta}_1} & \cdots & \sigma_{\Theta\tilde{\Delta}_K} \\ \sigma_{\tilde{\Delta}_1\mu} & \sigma_{\tilde{\Delta}_1\Theta} & \sigma_{\tilde{\Delta}_1}^2 & \cdots & \sigma_{\Delta_1\Delta_K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{\tilde{\Delta}_K\mu} & \sigma_{\tilde{\Delta}_K\Theta} & \sigma_{\tilde{\Delta}_K\tilde{\Delta}_1} & \cdots & \sigma_{\tilde{\Delta}_K}^2 \end{bmatrix}$$

where

$$\sigma_\mu^2 = \mathrm{Var}(y_i),$$

$$\begin{aligned}
\sigma_\Theta^2 &= 4\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j| - \Theta\big|y_i, g_i\big]^2\Big] \\
&= 4\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j|\big|y_i, g_i\big]^2\Big] - 4\Theta^2 \\
&= 4\mathbb{E}\bigg[\bigg((1-\pi_{g_i})\int_0^\infty |y_i - y| dF_{-k}(y)\bigg)^2\bigg] - 4\Theta^2 \\
&= 4\sum_{k=1}^K \pi_k \int_0^\infty \bigg((1-\pi_k)\int_0^\infty |x - y| dF_{-k}(y)\bigg)^2 dF_k(x) - 4\Theta^2 \\
&= 4\sum_{k=1}^K \pi_k (1-\pi_k)^2 \int_0^\infty \bigg(\int_0^\infty |x - y| dF_{-k}(y)\bigg)^2 dF_k(x) - 4\Theta^2,
\end{aligned}$$

$$\begin{aligned}
\sigma_{\tilde{\Delta}_k}^2 &= 4\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j| - \tilde{\Delta}_k\big|y_i, g_i\big]^2\Big] \\
&= 4\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j|\big|y_i, g_i\big]^2\Big] - 4\tilde{\Delta}_k^2 \\
&= 4\mathbb{E}\Big[\mathbb{1}_{\{g_i = k\}}\mathbb{E}\big[\mathbb{1}_{\{g_j = k\}}|y_i - y_j|\big|y_i, g_i\big]^2\Big] - 4\tilde{\Delta}_k^2 \\
&= 4\mathbb{E}\bigg[\mathbb{1}_{\{g_i = k\}}\bigg(\pi_k \int_0^\infty |y_i - y| dF_k(y)\bigg)^2\bigg] - 4\tilde{\Delta}_k^2 \\
&= 4\pi_k \int_0^\infty \bigg(\pi_k \int_0^\infty |x - y| dF_k(y)\bigg)^2 dF_k(x) - 4\tilde{\Delta}_k^2 \\
&= 4\pi_k^3 \int_0^\infty \bigg(\int_0^\infty |x - y| dF_k(y)\bigg)^2 dF_k(x) - 4\tilde{\Delta}_k^2,
\end{aligned}$$

$$\sigma_{\Theta,\mu} = 2\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j| - \Theta\big|y_i, g_i\big]\mathbb{E}\big[y_i - \mu\big|y_i, g_i\big]\Big]$$

$$
\begin{aligned}
&= 2\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j| - \Theta\big|y_i, g_i\big](y_i - \mu)\Big] \\
&= 2\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j| - \Theta\big|y_i, g_i\big]y_i\Big] - \underbrace{\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j| - \Theta\big|y_i, g_i\big]\mu\Big]}_{=0} \\
&= 2\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j|\big|y_i, g_i\big]y_i\Big] - 2\Theta\mu \\
&= 2\mathbb{E}\Big[(1 - \pi_{g_i})\int_0^\infty |y_i - y|dF_{-k}(y)y_i\Big] - 2\Theta\mu \\
&= 2\sum_{k=1}^K \pi_k(1 - \pi_k)\int_0^\infty \int_0^\infty |x - y|xdF_{-k}(y)dF_k(x) - 2\Theta\mu,
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{\tilde{\Delta}_k,\mu} &= 2\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j| - \tilde{\Delta}_k\big|y_i, g_i\big]\mathbb{E}\big[y_i - \mu\big|y_i, g_i\big]\Big] \\
&= 2\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j| - \tilde{\Delta}_k\big|y_i, g_i\big](y_i - \mu)\Big] \\
&= 2\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j| - \tilde{\Delta}_k\big|y_i, g_i\big]y_i\Big] \\
&\quad - \underbrace{2\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j| - \tilde{\Delta}_k\big|y_i, g_i\big]\mu\Big]}_{=0} \\
&= 2\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j|\big|y_i, g_i\big]y_i - \tilde{\Delta}_k y_i\Big] \\
&= 2\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j|\big|y_i, g_i\big]y_i\Big] - 2\tilde{\Delta}_k\mu \\
&= 2\mathbb{E}\Big[\mathbb{1}_{\{g_i = k\}}\mathbb{E}\big[\mathbb{1}_{\{g_j = k\}}|y_i - y_j|\big|y_i, g_i\big]y_i\Big] - 2\tilde{\Delta}_k\mu \\
&= 2\mathbb{E}\Big[\mathbb{1}_{\{g_i = k\}}\pi_k\int_0^\infty |y_i - y|dF_k(y)y_i\Big] - 2\tilde{\Delta}_k\mu \\
&= 2\pi_k\int_0^\infty \pi_k\int_0^\infty |x - y|dF_k(y)xdF_k(x) - 2\tilde{\Delta}_k\mu \\
&= 2\pi_k^2\int_0^\infty \int_0^\infty |x - y|xdF_k(y)dF_k(x) - 2\tilde{\Delta}_k\mu,
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{\Theta,\tilde{\Delta}_k} &= 4\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j| - \Theta\big|y_i, g_i\big]\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j| - \tilde{\Delta}_k\big|y_i, g_i\big]\Big] \\
&= 4\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j|\big|y_i, g_i\big]\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j|\big|y_i, g_i\big] \\
&\qquad - \tilde{\Delta}_k\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j|\big|y_i, g_i\big] - \Theta\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j|\big|y_i, g_i\big] + \Theta\tilde{\Delta}_k\Big] \\
&= 4\Big(\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j|\big|y_i, g_i\big]\mathbb{E}\big[\mathbb{1}_{\{g_i = k\}}\mathbb{1}_{\{g_j = k\}}|y_i - y_j|\big|y_i, g_i\big]\Big] \\
&\qquad - \Theta\tilde{\Delta}_k - \Theta\tilde{\Delta}_k + \Theta\tilde{\Delta}_k\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= 4\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i \neq g_j\}}|y_i - y_j|\big|y_i, g_i\big]\mathbb{E}\big[\mathbb{1}_{\{g_i=k\}}\mathbb{1}_{\{g_j=k\}}|y_i - y_j|\big|y_i, g_i\big]\Big] - 4\Theta\tilde{\Delta}_k \\
&= 4\mathbb{E}\Big[(1-\pi_{g_i})\int_0^\infty |y_i - y|dF_{-g_i}(y)\mathbb{1}_{\{g_i=k\}}\pi_k \int_0^\infty |y_i - y|dF_k(y)\Big] - 4\Theta\tilde{\Delta}_k \\
&= 4\pi_k \int_0^\infty (1-\pi_k)\int_0^\infty |x-y|dF_{-k}(y)\pi_k \int_0^\infty |x-y|dF_k(y)dF_k(x) - 4\Theta\tilde{\Delta}_k \\
&= 4\pi_k^2(1-\pi_k)\int_0^\infty \int_0^\infty |x-y|dF_{-k}(y)\int_0^\infty |x-y|dF_k(y)dF_k(x) - 4\Theta\tilde{\Delta}_k,
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{\tilde{\Delta}_k,\tilde{\Delta}_l} &= 4\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i=k\}}\mathbb{1}_{\{g_j=k\}}|y_i - y_j| - \tilde{\Delta}_k\big|y_i, g_i\big]\mathbb{E}\big[\mathbb{1}_{\{g_i=l\}}\mathbb{1}_{\{g_j=l\}}|y_i - y_j| - \tilde{\Delta}_l\big|y_i, g_i\big]\Big] \\
&= 4\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i=k\}}\mathbb{1}_{\{g_j=k\}}|y_i - y_j|\big|y_i, g_i\big]\mathbb{E}\big[\mathbb{1}_{\{g_i=l\}}\mathbb{1}_{\{g_j=l\}}|y_i - y_j|\big|y_i, g_i\big] \\
&\quad -\tilde{\Delta}_k\mathbb{E}\big[\mathbb{1}_{\{g_i=l\}}\mathbb{1}_{\{g_j=l\}}|y_i - y_j|\big|y_i, g_i\big] - \mathbb{E}\big[\mathbb{1}_{\{g_i=k\}}\mathbb{1}_{\{g_j=k\}}|y_i - y_j|\big|y_i, g_i\big]\tilde{\Delta}_l + \tilde{\Delta}_k\tilde{\Delta}_l\Big] \\
&= 4\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{g_i=k\}}\mathbb{1}_{\{g_j=k\}}|y_i - y_j|\big|y_i, g_i\big]\mathbb{E}\big[\mathbb{1}_{\{g_i=l\}}\mathbb{1}_{\{g_j=l\}}|y_i - y_j|\big|y_i, g_i\big] \\
&\quad - \tilde{\Delta}_k\tilde{\Delta}_l - \tilde{\Delta}_k\tilde{\Delta}_l + \tilde{\Delta}_k\tilde{\Delta}_l \\
&= -4\tilde{\Delta}_k\tilde{\Delta}_l
\end{aligned}
$$

Since $\frac{\partial G^W}{\partial \mu} = -\frac{G^W}{\mu}$ and $\frac{\partial G^W}{\partial \tilde{\Delta}_k} = \sqrt{\frac{G^W}{2\mu\tilde{\Delta}_k}}$ are both continuous functions and nonzero for $\mu \neq 0$ and $\tilde{\Delta}_k \neq 0$ for all $k = 1, \ldots, K$, we get from the delta method that G^W is asymptotically normally distributed with

$$
\sigma_W^2 = \left(\frac{G^W}{\mu}\right)^2 \mathrm{Var}(y_i) + \sum_{k=1}^K \frac{G^W}{2\mu\tilde{\Delta}_k}\sigma^2_{\tilde{\Delta}_k} - \sum_{k=1}^K \left(\frac{G^W}{\mu}\right)^{\frac{3}{2}} \sqrt{\frac{2}{\tilde{\Delta}_k}}\sigma_{\tilde{\Delta}_k,\mu} - \sum_{k\neq l} \frac{2G^W}{\mu}\sqrt{\tilde{\Delta}_k\tilde{\Delta}_l}.
$$

Similarly, since

$$
\begin{aligned}
\frac{\partial G^B}{\partial \mu} &= -\frac{G^B}{\mu} \\
\frac{\partial G^B}{\partial \Theta} &= \frac{1}{2\mu} \\
\frac{\partial G^B}{\partial \tilde{\Delta}_k} &= -\frac{1}{2\mu}\frac{\sum_{l\neq k}\sqrt{\tilde{\Delta}_l}}{\sqrt{\tilde{\Delta}_k}}
\end{aligned}
$$

are continuous functions and nonzero for $\mu \neq 0$, $\Theta \neq 0$, and $\tilde{\Delta}_k \neq 0$ for all $k = 1, \ldots, K$, we get from the delta method that G^B is asymptotically normally distributed with

$$\begin{aligned}\sigma_B^2 =&\left(\frac{G^B}{\mu}\right)^2 \mathrm{Var}(y_i) + \frac{1}{4\mu^2}\sigma_\Theta^2 + \sum_{k=1}^{K} \frac{1}{4\mu^2}\left(\frac{\sum_{l\neq k}\sqrt{\tilde{\Delta}_l}}{\sqrt{\tilde{\Delta}_k}}\right)^2 \sigma_{\tilde{\Delta}_k}^2 - \frac{G^B}{\mu^2}\sigma_{\Theta,\mu}\\
&- \sum_{k=1}^{K} \frac{1}{2\mu^2}\frac{\sum_{l\neq k}\sqrt{\tilde{\Delta}_l}}{\sqrt{\tilde{\Delta}_k}}\sigma_{\Theta,\tilde{\Delta}_k} + \sum_{k=1}^{K}\frac{G^B}{\mu^2}\frac{\sum_{l\neq k}\sqrt{\tilde{\Delta}_l}}{\sqrt{\tilde{\Delta}_k}}\sigma_{\mu,\tilde{\Delta}_k}\\
&- \sum_{k\neq l}\frac{1}{\mu^2}\left(\sum_{i\neq k}\sqrt{\tilde{\Delta}_i}\right)\left(\sum_{j\neq l}\sqrt{\tilde{\Delta}_j}\right)\sqrt{\tilde{\Delta}_k\tilde{\Delta}_l}\end{aligned}$$

Moreover, since the aggregate Gini coefficient is given by $G = \Delta/2\mu$, we get from the delta method that G is asymptotically normally distributed with asymptotic variance

$$\sigma_G^2 = \left(\frac{G}{\mu}\right)^2 \mathrm{Var}(y_i) + \frac{1}{4\mu^2}\sigma_\Delta - \frac{G}{\mu^2}\sigma_{\Delta\mu},$$

where

$$\begin{aligned}\sigma_\Delta^2 &= 4\int_0^\infty\int_0^\infty |x-y|^2 dF(y)dF(x) - 4\Delta^2\\
\sigma_{\Delta,\mu} &= 2\int_0^\infty\int_0^\infty |x-y|x dF(y)dF(x) - 2\Delta\mu.\end{aligned}$$

Finally, the delta method implies that $\alpha G^W + \beta G^B$ is asymptotically normally distributed for arbitrary constants α and β. Thus, G^W and G^B are asymptotically jointly normally distributed. Moreover, since $\mathrm{Var}(G) = \mathrm{Var}(G^W + G^B) = \mathrm{Var}(G^W) + \mathrm{Var}(G^B) + 2\mathrm{Cov}(G^W, G^B)$ or $\mathrm{Cov}(G^W, G^B) = \frac{1}{2}\big(\mathrm{Var}(G) - \mathrm{Var}(G^W) - \mathrm{Var}(G^B)\big)$, we get that

$$\sigma_{WB} = \frac{1}{2}\big(\sigma_G^2 - \sigma_W^2 - \sigma_B^2\big).$$

$\square$

A corollary of Theorem 3 is that the *shares* of within- and between-group inequality in aggregate inequality are also asymptotically normally distributed. Let $\alpha_W = G^W/G$ and $\alpha_B = G^B/G$ denote the share of within-group term and between-group term in aggregate inequality, respectively, and let $\alpha_{W,n}$ and $\alpha_{B,n}$ denote their values computed in a sample of size n. Corollary 4 states their asymptotic variances.

Corollary 4. *Under the conditions of Theorem 3, we have*

$$\sqrt{n}\big(\alpha_{W,n} - \alpha_W, \alpha_{B,n} - \alpha_B\big) \xrightarrow{d} \mathcal{N}(0, \Sigma),$$

where

$$\Sigma = \begin{bmatrix} \sigma_\alpha^2 & -\sigma_\alpha^2 \\ -\sigma_\alpha^2 & \sigma_\alpha^2 \end{bmatrix},$$

and

$$\sigma_\alpha^2 = \frac{(G^B)^2\sigma_W^2 + (G^W)^2\sigma_B^2 - 2G^W G^B \sigma_{WB}}{G^4},$$

where σ_W^2*,* σ_B^2*, and* σ_{WB} *are given in* (19)*,* (20)*, and* (21)*, respectively.*

Proof. Since $\alpha_W = G^W/(G^W + G^B)$, where G^W and G^B are asymptotically jointly normally distributed, and

$$\frac{\partial \alpha_W}{\partial G^W} = \frac{1 - \alpha_W}{G},$$
$$\frac{\partial \alpha_W}{\partial G^B} = -\frac{\alpha_W}{G},$$

which are continuous nonzero functions, by the delta method we get that α_W is asymptotically normally distributed with

$$\sigma_\alpha^2 = \frac{(G^B)^2\sigma_W^2 + (G^W)^2\sigma_B^2 - 2G^W G^B \sigma_{WB}}{G^4}.$$

Asymptotic normality and variance of α_B are shown similarly. Finally, since

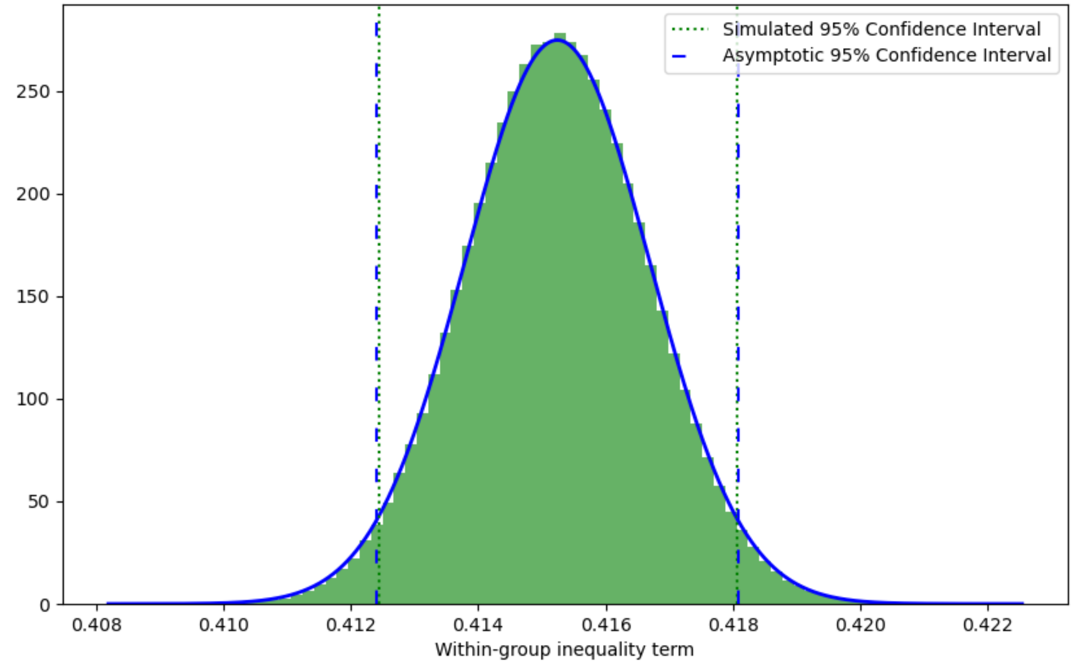
$$a\alpha_W + b\alpha_B = \frac{aG^W + bG^B}{G^W + G^B}$$

which is a continuously differentiable function in G^W and G^B for arbitrary constants a, b and since G^W and G^B are asymptotically jointly normally distributed, then α_W and α_B are asymptotically jointly normally distributed by the delta method. Moreover, since $Var(\alpha_W + \alpha_B) = Var(\alpha_W) + Var(\alpha_B) + 2Cov(\alpha_W, \alpha_B) = 2\sigma_\alpha^2 + 2Cov(\alpha_W, \alpha_B) = 0$, we get $Cov(\alpha_W, \alpha_B) = -\sigma_\alpha^2$.

□

B.2 Simulation

We confirm through simulation exercises that inference based on the asymptotic variances derived above yields very precise confidence intervals for lognormally distributed incomes and sample sizes similar to those in our empirical application. Specifically, we compare the asymptotic confidence intervals to simulated confidence intervals where we assume that incomes in each subgroup are log-normally distributed with means and Gini coefficients equal to their sample equivalents and we repeatedly draw samples of the same size as the samples in our empirical application of section V. Figure 3 plots the distribution of the sampled within and between-group inequality terms against a (centered) normal distribution where the variance is derived using the asymptotic result from Theorem 3. The figure confirms that the formulas for the asymptotic variances produce very precise confidence intervals for lognormally distributed data and sample sizes similar to those in our empirical application.



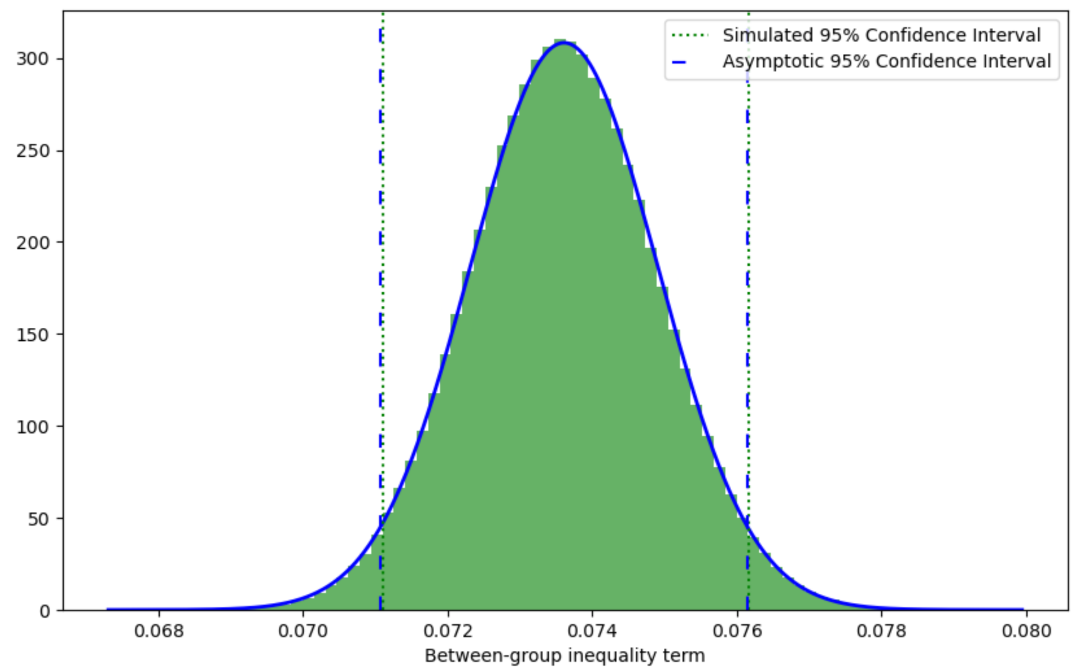


Figure 3: Asymptotic vs. simulated confidence intervals. We draw i.i.d samples from lognormal distributions with means and Gini coefficients equal to the observed average incomes and Gini coefficients of the demographic subgroups in the year 2021. We further set the probability of sampling from any given demographic subgroup equal to its observed population share in the same year, and we draw repeated samples with total size equal to the number of observations in our empirical application (58,101 observations).

C Subgroup Decompositions of Multivariate Gini Coefficients

Koshevoy and Mosler (1997) suggest extensions of the Gini coefficient to multivariate distributions based on the volume of Lorenz zonoids, which are multivariate extensions of Lorenz regions. The multivariate Gini coefficient for a d-dimensional distribution is given by the following expression,

$$G^d = \frac{1}{2^d - 1} \left| Z^{d+1} \right|, \tag{22}$$

where Z^{d+1} is the Lorenz zonoid and the scaling factor $1/(2^d - 1)$ ensures that the Gini coefficient for distributions with non-negative support is contained in $[0, 1]$.[7]

As with Lorenz regions for univariate distributions that we discus in section IV.3 of the main text, in the case of multivariate distributions we can also define a subgroup Lorenz zonoid, Z_k^{d+1}, for subgroup k by scaling the subgroup's Lorenz zonoid with a vector $(\pi_k, \theta_{k,1}, \ldots, \theta_{k,d})$ where π_k is the subgroup's population share and $\theta_{k,j}$ is the subgroup's share of attribute j. Because the aggregate Lorenz zonoid can be computed as a Minkowski sum of the subgroup Lorenz zonoids (Mosler (2002)), the Gini decomposition of Theorem 2 extends naturally to the multivariate case. That is, within-group inequality, G^W, can be defined as the minimum volume of the aggregate Lorenz zonoid for given volumes of the subgroup Lorenz zonoids, scaled by the scaling factor $1/(2^d - 1)$. By the Brunn-Minkowski theorem, the minimum volume of the scaled aggregate Lorenz zonoid is

$$G^W = \frac{1}{2^d - 1} \left(\sum_{k=1}^{K} \left| Z_k^{d+1} \right|^{\frac{1}{d+1}} \right)^{d+1} = \left(\sum_{k=1}^{K} \left(\pi_k \prod_{j=1}^{d} \theta_{k,j} G_k^d \right)^{\frac{1}{d+1}} \right)^{d+1}.$$

As in the univariate case, between-group inequality can be computed as the difference between the aggregate Gini coefficient and the within-group inequality term.

Koshevoy and Mosler (1997) point out that the multivariate Gini coefficient in (22) is zero when there is no inequality in at least one of the attributes. This means that the Gini coefficient can be zero even if there is inequality along some, but not all, dimensions. Because this may be seen as an undesirable property, Koshevoy and

[7] The Lorenz zonoid for a d-dimensional distribution with cumulative distribution function F is given by $Z^{d+1}(F) = \{(z_0(\tilde{F}, h), z(\tilde{F}, h)),\ h : \mathbb{R}^d \to [0, 1] \text{ measurable}\}$, where $z_0(\tilde{F}, h) = \int_{\mathbb{R}^d} h(x) d\tilde{F}(x)$, $z(\tilde{F}, h) = \int_{\mathbb{R}^d} h(x) x d\tilde{F}(x)$, and $\tilde{F}$ is obtained by component-wise scaling F by its mean vector.

Mosler (1997) also discuss an alternative multivariate Gini coefficient that is zero only if there is no inequality in all dimensions. This alternative Gini coefficient is given by the following expression,

$$\tilde{G}^d = \frac{1}{2^d - 1}\left(\left|Z^{d+1} \oplus C^d\right| - 1\right), \tag{23}$$

where Z^{d+1} is again the Lorenz zonoid and C^d is the d-dimensional unit cube. $Z^{d+1} \oplus C^d$ is referred to as the *expanded* Lorenz zonoid. Similarly, one can define expanded subgroup Lorenz zonoids as $Z_k^{d+1} \oplus C_k^d$, where Z_k^{d+1} is defined as above and C_k^d is the d-dimensional unit cube scaled by the vector $(\pi_k, \theta_{k,1}, \ldots, \theta_{k,d})$.

It can easily be shown that the expanded Lorenz zonoid for the aggregate distribution is the Minkowski sum of the expanded subgroup Lorenz zonoids. Therefore, one can define within-group inequality analogously for this Gini coefficient as the minimum volume of the expanded aggregate Lorenz zonoid for given volumes of the expanded subgroup Lorenz zonoids:

$$\begin{aligned}
\tilde{G}^W &= \frac{1}{2^d - 1}\left(\left(\sum_{k=1}^{K}\left(\left|Z_k^{d+1} \oplus C_k^d\right|\right)^{\frac{1}{d+1}}\right)^{d+1} - 1\right) \\
&= \frac{1}{2^d - 1}\left(\left(\sum_{k=1}^{K}\left(\pi_k \prod_{j=1}^{d} \theta_{k,j}\left(\left(2^d - 1\right)\tilde{G}_k + 1\right)\right)^{\frac{1}{d+1}}\right)^{d+1} - 1\right)
\end{aligned}$$

Note that, just like within-group inequality in the univariate case, G^W and $\tilde{G}^W$ are weighted quasi-arithmetic means of the subgroup Gini coefficients with weights that depend on population and income shares and do not necessarily sum up to one.

D Arithmetic Interpretation

For the arithmetic interpretation of the Gini decomposition, we consider a finite population. The Gini coefficient can then be defined arithmetically as the sum of absolute income differences between all possible pairs of individuals in a population P, normalized by twice the product of the average income and the total number of such pairs:

$$G = \frac{\sum_{i,j \in P} |y_i - y_j|}{2\mu N^2}. \tag{24}$$

Proposition 13 states that the within-group inequality term in our Gini decomposition is equal to the minimal sum of absolute differences in the population for given sums of absolute differences in each subgroup, again normalized by $2\mu N^2$.

Proposition 13. *Let S_k be a collection of real numbers for each $k \in \{1, \ldots, K\}$ and $S = \bigcup_{k=1}^{K} S_k$. Then, the following inequality holds:*

$$G = \frac{1}{2\mu N^2} \sum_{x,y \in S} |x - y| \geq \frac{1}{2\mu N^2} \left(\sum_{k=1}^{K} \sqrt{\sum_{x,y \in S_k} |x - y|} \right)^2 = G^W, \tag{25}$$

where μ and N are the mean and the cardinality of S. Equality holds if and only if the distribution of numbers in each collection S_k is identical.

Proof. Let $G(S_k)$ denote the Gini coefficient computed on the collection of real numbers S_k and let S denote the union of S_k. Now inequality (25) follows from

$$\frac{1}{2\mu N^2} \sum_{x,y \in S} |x - y| = G \geq G^W = \left(\sum_{k=1}^{K} \sqrt{\pi_k \theta_k G(S_k)} \right)^2 = \frac{1}{2\mu N^2} \left(\sum_{k=1}^{K} \sqrt{2\mu_k N_k^2 G(S_k)} \right)^2$$
$$= \frac{1}{2\mu N^2} \left(\sum_{k=1}^{K} \sqrt{\sum_{x,y \in S_k} |x - y|} \right)^2,$$

where μ_k and N_k are the mean and the cardinality of S_k, respectively, and μ and N are the mean and cardinality of S, and $\pi_k = N_k/N$ and $\theta_k = (N_k \mu_k)/(N\mu)$. The inequality above follows directly from Proposition 8.

□

This result allows for an insightful comparison between our decomposition formula and the formula by Bhattacharya and Mahalanobis (1967). In that formula within-group inequality is equal to

$$\frac{\sum_{k=1}^{K} \sum_{i,j \in k} |y_i - y_j|}{2\mu N^2}, \tag{26}$$

that is, within-group inequality is equal to the sum of absolute differences between incomes in all pairs where both individuals belong to the same subgroup, normalized by $2\mu N^2$. Whereas absolute differences between incomes of individuals from the same subgroup clearly contribute to within-group inequality, this expression has the undesirable property that within-group inequality is less than aggregate inequality even if all subgroups have identical distributions. Thus, within-group inequality must be be larger than (26). Proposition 13 implies that our within-group inequality term is always greater than (26) and is equal to aggregate inequality when all subgroups have identical income distributions.

E Illustration with Continuous Distributions

In Figure 1 of the main paper, we illustrate the aggregation of subgroup Lorenz regions when income distributions are discrete. Figure 4 below illustrates the aggregation of subgroup Lorenz regions in the case of continuous distributions. In this example, there are two subgroups with Pareto income distributions. The subgroup Lorenz regions are shown in the left and the middle panel. In the right panel, the aggregate Lorenz region corresponding to the two subgroup Lorenz regions is outlined by the solid lines. As shown in Proposition 7 of the main paper, the aggregate Lorenz region is equal to the Minkowski sum of the subgroup Lorenz regions.

Figure 4 also illustrates within-group and between-group inequality. The dashed lines in the rightmost panel outline the aggregate Lorenz region in a counterfactual case where the subgroup Lorenz regions have the same areas as before but are homothetic.[8] Thus, the area of the dashed aggregate Lorenz region equals within-group inequality and the excess area in the solid aggregate Lorenz region (the shaded area) equals between-group inequality. As discussed in section IV.3 of the main paper, between-group inequality results from the subgroup Lorenz regions having different shapes.

F Additional Empirical Applications

In this appendix, we present additional empirical applications and make use of the fact that lump-sum transfers between subgroups do not affect within-group inequality (Proposition 5) to further isolate the part of between-group inequality that remains after differences in average incomes between subgroups are removed.

[8]Specifically, we consider a counterfactual case where the subgroup income distributions are Pareto with common parameter values, while the subgroup population shares and areas of the subgroup Lorenz regions stay the same. Note that making subgroup Lorenz regions homothetic while preserving their areas may require changing the population shares of the subgroups.

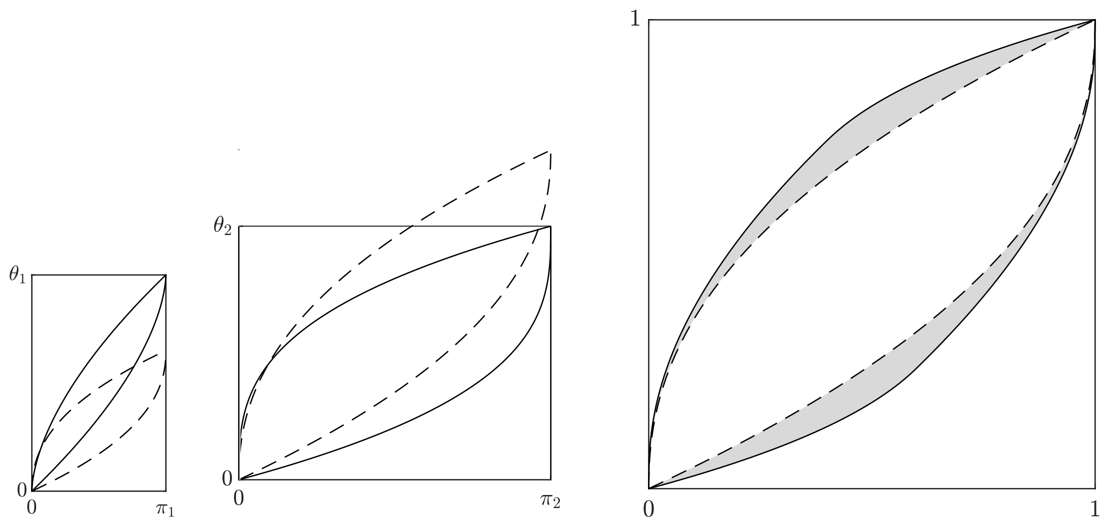


Figure 4: Illustration of the aggregation of continuous income distributions and between-group inequality. The left and middle panel show subgroup Lorenz regions, while the right panel shows the corresponding aggregate Lorenz regions. The solid lines outline the subgroup Lorenz regions of two different Pareto distributions and the Lorenz region of the corresponding aggregate distribution. The dashed lines outline counterfactual subgroup Lorenz regions that have the same area as the original subgroup Lorenz regions but are homothetic, i.e., representing the same distribution (again Pareto), and the Lorenz region of the corresponding aggregate distribution. The area of the counterfactual aggregate Lorenz region is equal to within-group inequality and the shaded area is equal to between-group inequality.

F.1 Income inequality within and between demographic subgroups

We first revisit the empirical application of section V, and use Proposition 5 to isolate the role of differences in subgroup average incomes. Figure 5 shows that while between-group inequality has remained roughly constant, differences in average incomes between demographic subgroups now play a smaller role in between-group inequality compared to the beginning of the sample period. The between-group inequality that remains after the means were equalized reflects the contribution to aggregate inequality stemming from differences between subgroups in higher-order moments of the income distribution.

F.2 Gender Earnings Inequality

In this exercise, we apply the subgroup decomposition of the Gini coefficient to study how differences in male versus female earnings distributions contribute to overall earnings inequality in the United States.

We use data from the Current Population Surveys on earnings of people aged 18-64,

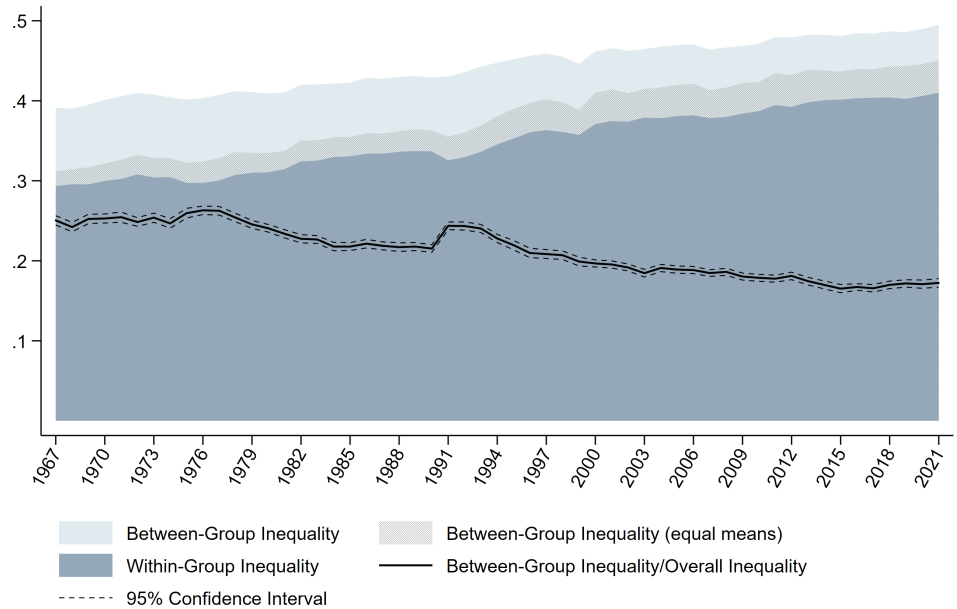


Figure 5: Household-level income inequality within and between demographic subgroups, 1967–2021. The hatched region shows between-group inequality when differences in average earnings between demographic subgroups are removed by implementing suitable lump-sump transfers. The demographic subgroups are defined as in the main text. Asymptotic confidence intervals are computed using the formulas derived in online appendix B.

working full-time, and earning more than 10% of the median, and we group individuals by sex. Figure 6 shows the evolution of within-group and between-group inequality since the beginning of our sample period. We document a strong decline in the share of the aggregate Gini coefficient that can be attributed to differences between the earnings distributions of men and women (from 12% in 1967 to 1% in 2021). While between-group inequality has declined over time, within-group inequality has increased by more so that aggregate earnings inequality is now significantly higher than half a century ago.

Making use of the fact that lump-sum transfers between subgroups do not affect within-group inequality, we further isolate the part of between-group inequality that remains after differences in average earnings between men and women are removed. While differences in average earnings contributed significantly to the level of aggregate inequality early in the sample period, eliminating differences in average earnings via lump-sum transfers hardly reduces the Gini coefficient in recent years, with the remaining between-group inequality reflecting differences in higher-order moments of the

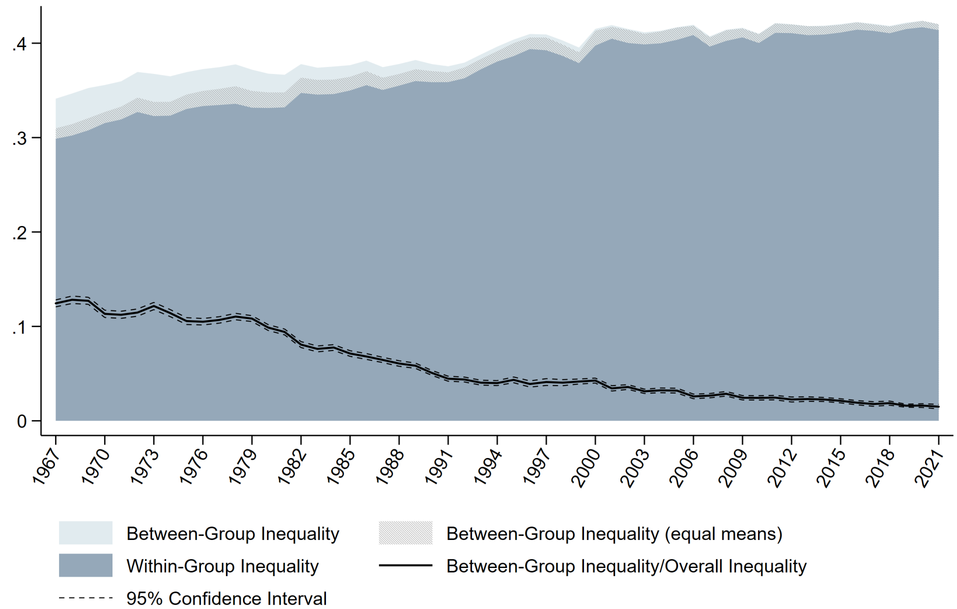


Figure 6: Earnings inequality within and between sexes in the United States, 1967–2021. The hatched region shows between-group inequality when sex differences in average earnings are removed by implementing suitable lump-sump transfers. Asymptotic confidence intervals are computed using the formulas derived in online appendix B.

male and female earnings distributions.[9] This finding does not mean that the average earnings of men and women have fully converged. In fact, average earnings of women in 2021 remain at only 78% of those of men. These differences in means, however, are small compared to the large variation in earnings within each gender group and therefore contribute little to aggregate inequality.

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[9]Beyond the difference in means, the male and female earning distributions in the latest survey year also differ in their variances (94% higher in the male relative to the female earnings distribution) and skewness (29% higher for the female relative to the male earnings distribution. The between-group inequality that remains after the means were equalized reflects the contribution to aggregate inequality stemming from gender differences across all higher-order moments of the income distribution.

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